Solutions to the Mathematics 1571 Final Exam Review Problems

1. Consider the function f defined by $f(x) = 2x^2 - 5x$. Determine the following:

   (a) $f(a+b)$ (b) $f(2x) - 2f(x)$

   Solution:

   (a) Substituting in $(a+b)$ for x, gives

   $$\begin{aligned} f(a+b) =& 2(a+b)^2 - 5(a+b) \\ =& 2(a^2 + 2ab + b^2) - 5(a+b) \\ =& 2a^2 + 4ab + 2b^2 - 5a - 5b \end{aligned}$$

   (b) Again, substituting in appropriate quantities for x, gives

   $$\begin{aligned} f(2x) - 2f(x) =& 2(2x)^2 - 5(2x) - 2(2x^2 - 5x) \\ =& 2 \cdot 4x^2 - 10x - \left[4x^2 - 10x\right] \\ =& 8x^2 - 4x^2 - 10x + 10x \\ =& 4x^2 \end{aligned}$$

2. Find the domain of g if

   (a) $g(x) = \sqrt{x^2 - 3x - 4}$ (b) $g(x) = \dfrac{x+2}{x^3 - x}$

   Solution:

   (a) The domain will be the set of real numbers such that $x^2 - 3x - 4 \geq 0$. Factoring, this gives

   $$(x-4)(x+1) \geq 0$$

   One way to solve this inequality is to use sign analysis.

$x-4$	$----$	$----$	$++++$
$x+1$	$----$	$++++$	$++++$
Product	$++++$	$----$	$++++$
		-1	4

So the domain is given by $x \geq 4$ or $x \leq -1$. In interval notation this set is $(-\infty, 1] \cup [4, \infty)$.

(b) First factor the denominator

$$g(x) = \frac{x+2}{x^3 - x} = \frac{x+2}{x(x^2-1)} = \frac{x+2}{x(x+1)(x-1)}.$$

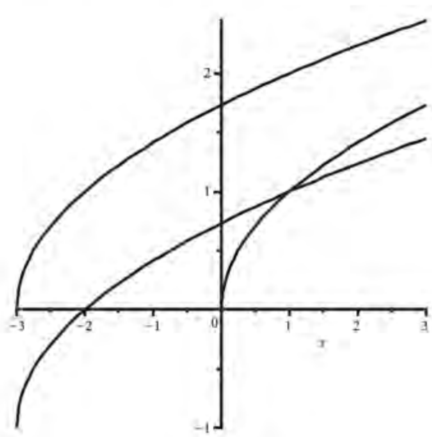
The domain is all real numbers except those for which the denominator is zero. The denominator will be zero when :

$$x = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$

that is, when $x = 0$ or $x = -1$ or $x = +1$
In interval notation, the domain is $(-\infty, -1) \cup (-1, 0) \cup (0, 1)$.

3. The graph of $f(x) = \sqrt{x+3} - 1$ is the same as the graph of $f(x) = \sqrt{x}$ except that it is translated in what way?

Solution:
Since the one is negative, the graph is moved downward 1 unit, and since the 3 is positive, it is moved 3 units to the left.



4. For the functions f and g, find $f \circ g$, $g \circ f$, and the domains of each.

(a) $f(x) = \dfrac{1}{x^2 - 1}$ and $g(x) = \sqrt{x}$ (b) $f(x) = x^2 + 1$ and $g(x) = \dfrac{x}{x-1}$

Solution:

(a) $f(x) = \dfrac{1}{x^2 - 1}$, $g(x) = \sqrt{x}$

$(f \circ g)(x) = f(g(x)) = \dfrac{1}{(g(x))^2 - 1} = \dfrac{1}{(\sqrt{x})^2 - 1} = \dfrac{1}{x-1}$

To find the domain of $f \circ g$:

- $f \circ g$ is not defined where g is not defined.
- $f \circ g$ is not defined where f is not defined at the value g takes at x.

So: $g(x)$ is not defined for $x < 0$ [eliminate $x < 0$ from domain] and the denominator of f is not defined at $x = \pm 1$, that is $f \circ g$ is not defined at $g(x) = \pm 1$, that is, at $x = 1$ (since $\sqrt{x}$ is never negative).

The range is given by $x \geq 0, x \neq 1$, that is, $[0, 1) \cup (1, \infty)$

$$g(x) = \sqrt{x}\ ,\ f(x) = \frac{1}{x^2 - 1}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g\left(\frac{1}{x^2 - 1}\right) \\ &= \sqrt{\frac{1}{x^2 - 1}}\end{aligned}$$

For the domain, we must have $\dfrac{1}{x^2 - 1} \geq 0$, so we must have $x^2 - 1 \geq 0$.

$x^2 - 1 = (x + 1)(x - 1)$

Sign analysis:

	$(-\infty, -1)$	$(-1, 1)$	$(1, \infty)$
$x-1$	$- - - -$	$- - - -$	$+ + + +$
$x+1$	$- - - -$	$+ + + +$	$+ + + +$
x^2-1	$+ + + +$	$- - - -$	$+ + + +$

$x^2 - 1 \geq 0$ exactly when $x \leq -1$ or $x \geq +1$, that is, $|x| \geq 1$.
Also, $x^2 - 1$ must not be zero, so $x \neq \pm 1$. So, $|x| > 1$.
Domain: $(-\infty, -1) \cup (1, \infty)$.

(b) $f(x) = x^2 + 1$, $g(x) = \dfrac{x}{x-1}$.

$$
\begin{aligned}
(f \circ g)(x) &= f\,(g(x)) \\
&= f\left(\frac{x}{x-1}\right) \\
&= \left(\frac{x}{x-1}\right)^2 + 1 \\
&= \frac{x^2}{(x-1)^2} + 1 \\
&= \frac{x^2}{(x-1)^2} + \frac{(x-1)^2}{(x-1)^2} \\
&= \frac{x^2 + (x-1)^2}{(x-1)^2} \\
&= \frac{x^2 + x^2 - 2x + 1}{(x-1)^2} \\
&= \frac{2x^2 - 2x + 1}{(x-1)^2}
\end{aligned}
$$

The domain will be all real numbers x such that $x \neq 1$, that is $(-\infty, 1) \cup (1, \infty)$.

$$\{x \in \mathbb{R} | x \neq 1\}$$

$$
\begin{aligned}
(g \circ f)(x) &= g\,(f(x)) \\
&= g(x^2 + 1) \\
&= \frac{x^2 + 1}{x^2 + 1 - 1} = \frac{x^2 + 1}{x^2}
\end{aligned}
$$

Domain: All real numbers such that $x \neq 0$, that is $(-\infty, 0) \cup (0, \infty)$.

$$\{x \in \mathbb{R} | x \neq 0\}$$

5. Find the intercepts of the following equations. Also determine whether the equations are symmetric with respect to the y-axis or the origin.

(a) $y = x^4 + x^3 + x^2$

(b) $y = \dfrac{1}{x^3 - 3x}$

(c) $y = 2 - |x|$

Solution:

(a) **y-intercepts:**
Let $x = 0$. Then $y = 0^4 + 0^3 + 0^2 = 0$.
y-intercept is 0.

x-intercepts:

$$\begin{aligned} y &= 0 \\ x^4 + x^3 + x^2 &= 0 \\ x^2(x^2 + x + 1) &= 0 \\ x^2 = 0 &\text{ or } x^2 + x + 1 = 0 \end{aligned}$$

$x = 0$ or by quadratic formula $x = \dfrac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2}$
$x = 0$ is only x-intercept.

Symmetry:
Substitute $-x$ for x:

$$\begin{aligned} (-x)^4 + (-x)^3 + (-x)^2 &= x^4 - x^3 + x^2 \\ &\neq x^4 + x^3 + x^2 \end{aligned}$$

So not symmetric about y-axis.

Origin:
$y = x^4 + x^3 + x^2$

Substitute $-x$ for x and $-y$ for y.

$$\begin{aligned} -y =&(-x)^4 + (-x)^3 + (-x)^2 \\ y =& -x^4 + x^3 - x^2 \end{aligned}$$

Not symmetric about origin.

(b) $y = \dfrac{1}{x^3 - 3x}$

y-intercepts:
y does not exist when $x = 0$, so there are no y-intercepts.
x-intercepts:
$0 = \frac{1}{x^3-3x}$ has no solutions. No intercepts.
Exchange $-x$ for x:

$$\begin{aligned} \frac{1}{(-x)^3 - 3(-x)} =& \frac{1}{-x^3 + 3x} \\ =& -\frac{1}{x^3 - 3x} \end{aligned}$$

Not symmetric with respect to y-axis.

(c) $y = 2 - |x|$

y-intercepts:
Let $x = 0$. Then $y = 2 - |0| = 2$
$(0, 2)$

x-intercepts:
Let $y = 0$. Then

$$\begin{aligned} 0 =& 2 - |x| \\ |x| =& 2 \\ x =& \pm 2 \end{aligned}$$

$(2, 0)$, $(-2, 0)$

Intercepts: $(0, 2)$, $(2, 0)$, $(-2, 0)$

Symmetry:

Check for symmetry with respect to y-axis:

$$2 - |-x| = 2 - |x|$$

Symmetric about y-axis.
Check for symmetry about origin:
Exchange x and y: $x = 2 - |y|$
This doesn't even give a function, so there is not symmetry about the origin.

6. Determine

(a) $\lim_{x\to 1} f(x)$, if $f(x) = \begin{cases} 4x - 2, & \text{if } x < 1 \\ 7, & \text{if } x = 1 \\ 2x - x^2, & \text{if } x > 1 \end{cases}$

(b) $\lim_{x\to -2} g(x)$, if $g(x) = \begin{cases} x + 5, & \text{if } x \geq -2 \\ 2x^2 - x - 7, & \text{if } x < -2 \end{cases}$

Solution:

(a) The limit does not depend on the value at $x = 1$. It exists *iff* $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$ both exist, and are equal.

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x - x^2 = 2 - 1 = 1$$
$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 4x - 2 = 4 - 2 = 2 \neq 1$$

$\lim_{x\to 1} f(x)$ does not exist.

(b) This limit exists *iff* both directed limits exist and are equal:

$$\lim_{x\to -2^-} g(x) = \lim_{x\to -2^-} 2x^2 - x - 7 = 2(-2)^2 - (-2) - 7 = 2 \cdot 4 + 2 - 7 = 3$$
$$\lim_{x\to -2^+} g(x) = \lim_{x\to -2^+} x + 5 = -2 + 5 = 3$$

So $\lim_{x\to -2} g(x) = 3$

7. Determine

(a) $\lim_{x\to 2} \left(\sqrt{x-1} - \sqrt{3x-2}\right)$ (b) $\lim_{x\to -1} \dfrac{x^3 + 1}{x^2 - 4x - 5}$

(c) $\lim_{x\to\infty} \frac{4x^3+x-5}{x^3-2}$

(d) $\lim_{x\to 7^-} \frac{x^2+49}{x-7}$

(e) $\lim_{x\to 7^+} \frac{x^2+49}{x-7}$

(f) $\lim_{x\to 7} \frac{x^2+49}{x-7}$

(g) $\lim_{x\to \frac{1}{2}^+} \frac{|2x-1|}{2x-1}$

(h) $\lim_{x\to \frac{1}{2}^-} \frac{|2x-1|}{2x-1}$

(i) $\lim_{x\to \frac{1}{2}} \frac{|2x-1|}{2x-1}$

(j) $\lim_{x\to -\infty} \frac{x^2+4x-1}{x^4}$

(k) $\lim_{x\to -\infty} \frac{\sqrt[3]{x^3+2x^2+1}}{3x-5}$

(l) $\lim_{x\to -3^+} \sqrt{x^2-9}$

(m) $\lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{x}$

(n) $\lim_{x\to \frac{\pi}{2}^-} \frac{\sec x - 1}{\tan x}$

(o) $\lim_{x\to 1} \frac{\sqrt{8+x^2}-3}{x-1}$

(p) $\lim_{h\to 0} \frac{\tan\left(\frac{\pi}{3}+h\right)-\tan\frac{\pi}{3}}{h}$

(q) $\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{6}+h\right)-\sin\left(\frac{\pi}{6}\right)}{h}$

(r) $\lim_{h\to 0} \frac{\cos(x+h)-\cos x}{h}$

Solution:

(a)

$$\begin{aligned}\lim_{x\to 2}\left(\sqrt{x-1}-\sqrt{3x-2}\right) &= \sqrt{2-1}-\sqrt{6-2}\\ &= \sqrt{1}-\sqrt{4}\\ &= 1-2\\ &= -1\end{aligned}$$

(b) $\lim_{x\to -1} \frac{x^3+1}{x^2-4x-5}$

Simply substituting $x=-1$ doesn't work. It yields $\frac{0}{0}$

$$\begin{aligned}\lim_{x\to -1} \frac{x^3+1}{x^2-4x-5} &= \lim_{x\to -1} \frac{(x+1)(x^2-x+1)}{(x+1)(x-5)}\\ &= \lim_{x\to -1} \frac{x^2-x+1}{x-5}\end{aligned}$$

$$= \frac{(-1)^2 - (-1) + 1}{-1 - 5} = \frac{3}{-6} = -\frac{1}{2}$$

(c) $\lim_{x \to \infty} \frac{4x^3 + x - 5}{x^3 - 2} = 4$

(d) $\lim_{x \to 7^-} \frac{x^2 + 49}{x - 7} = -\infty$

Substituting $x = 7$ yields $\frac{49 + 49}{0}$, but $x - 7$ approaches 0 from below as x approaches 7 from below, so answer is negative.

(e) $\lim_{x \to 7^+} \frac{x^2 + 49}{x - 7} = +\infty$ since substituting $x = 7$ yields form $\frac{49 + 49}{0}$ and denominator approaches 0 from above.

(f) $\lim_{x \to 7} \frac{x^2 + 49}{x - 7}$ does not exist, by combining results of (d) and (e).

(g) $\lim_{x \to \frac{1}{2}^+} \frac{|2x - 1|}{2x - 1} = +1$

Explanation: For $x > \frac{1}{2}$, $2x - 1$ is positive, so $|2x - 1| = 2x - 1$.

(h) $\lim_{x \to \frac{1}{2}^-} \frac{|2x - 1|}{2x - 1} = -1$

Explanation: For $x < \frac{1}{2}$, $2x - 1$ is negative, so $|2x - 1| = -(2x - 1)$.

(i) $\lim_{x \to \frac{1}{2}} \frac{|2x - 1|}{2x - 1}$ does not exist, by answers to (g) and (h).

(j) $\lim_{x \to -\infty} \frac{x^2 + 4x - 1}{x^4} = 0$

Explanation: Both leading terms are positive, x^4 dominates.

(k)

$$\lim_{x \to -\infty} \frac{\sqrt[3]{x^3 + 2x^2 + 1}}{3x - 5} = \lim_{x \to -\infty} \frac{\left(\sqrt[3]{x^3 + 2x^2 + 1}\right) \cdot \frac{1}{x}}{3 - \frac{5}{x}}$$
$$= \lim_{x \to -\infty} \frac{\left(\sqrt[3]{1 + \frac{2}{x} + \frac{1}{x^3}}\right)}{3 - \frac{5}{x}} = \frac{\sqrt[3]{1}}{3} = \frac{1}{3}$$

(l) $\lim_{x \to -3^+} \sqrt{x^2 - 9}$

At first glance, this may appear to be 0. However, note that for values slightly above

-3, $x^2 - 9$ is negative. Since the expression does not exist as x approaches -3 from above, neither does the limit.

(m) $\displaystyle \lim_{x\to\infty} \frac{\sin\left(\frac{1}{x}\right)}{x} = 0$

(Remark: As $x \to \infty$, $\frac{1}{x} \to 0$ from above, so $\sin\left(\frac{1}{x}\right) \to 0$ also. So this has form $\frac{0}{\infty}$, so answer is 0.)

(n) $\displaystyle \lim_{x\to\frac{\pi}{2}^-} \frac{\sec x - 1}{\tan x}$

$\sec \frac{\pi}{2}^- = \infty$

$\tan \frac{\pi}{2}^- =$undefined (∞)

Substituting $x = \frac{\pi}{2}$ gives $\frac{\infty}{\infty}$.

$$\begin{aligned}
\lim_{x\to\frac{\pi}{2}^-} \frac{\sec x - 1}{\tan x} &= \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{1}{\cos x} - 1}{\frac{\sin x}{\cos x}} \\
&= \lim_{x\to\frac{\pi}{2}^-} \frac{1 - \cos x}{\sin x} \\
&= \frac{1-0}{1} \\
&= 1
\end{aligned}$$

(o) $\displaystyle \lim_{x\to 1} \frac{\sqrt{8+x^2} - 3}{x - 1}$

Substituting $x = 1$ yields $\frac{0}{0}$:

$$\frac{\sqrt{8+1} - 3}{1-1} = \frac{\sqrt{9} - 3}{0} = \frac{0}{0}$$

Multiplying by the conjugate of the numerator gives

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{8+x^2} - 3}{x-1} &= \lim_{x\to 1} \frac{(8+x^2)^{1/2} - 3}{x-1} \cdot \frac{(8+x^2)^{1/2} + 3}{(8+x^2)^{1/2} + 3} \\
&= \lim_{x\to 1} \frac{(8+x^2) - 9}{(x-1)((8+x^2)^{1/2} + 3)} \\
&= \lim_{x\to 1} \frac{x^2 - 1}{(x-1)((8+x^2)^{1/2} + 3)} \\
&= \lim_{x\to 1} \frac{(x-1)(x+1)}{(x-1)((8+x^2)^{1/2} + 3)}
\end{aligned}$$

$$= \lim_{x \to 1} \frac{x+1}{(8+x^2)^{1/2}+3} = \frac{2}{6} = \frac{1}{3}$$

(p) $\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{3}+h\right) - \tan\frac{\pi}{3}}{h}$

The limit is the derivative of the tangent function at $\frac{\pi}{3}$. Hence,

$$\lim_{h \to 0} \frac{\tan\left(\frac{\pi}{3}+h\right) - \tan\frac{\pi}{3}}{h} = \left(\sec\left(\frac{\pi}{3}\right)\right)^2$$
$$= 4$$

(q) $\lim_{h \to 0} \frac{\sin\left(\frac{\pi}{6}+h\right) - \sin\left(\frac{\pi}{6}\right)}{h}$

The limit is the derivative of the sine function at $\frac{\pi}{6}$, so

$$\lim_{h \to 0} \frac{\sin\left(\frac{\pi}{6}+h\right) - \sin\left(\frac{\pi}{6}\right)}{h} = \cos\left(\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{2}$$

(r) $\lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$

The limit is the derivative of the cosine function , so

$$\lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h} = -\sin(x)$$

8. Determine P such that the following functions are continuous at $x = 3$.

(a) $f(x) = \begin{cases} \frac{3(x^4-81)}{x^2-9}, & \text{if } x \neq 3 \\ Px+9, & \text{if } x = 3 \end{cases}$

(b) $g(x) = \begin{cases} \frac{x^3-27}{x^2-9}, & \text{if } x \neq 3 \\ P, & \text{if } x = 3 \end{cases}$

Solution:

(a) $f(x) = \begin{cases} \frac{3(x^4-81)}{x^2-9}, & \text{if } x \neq 3 \\ Px+9, & \text{if } x = 3 \end{cases}$

Factoring the top expression:

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{3(x^2-9)(x^2+9)}{x^2-9} = \lim_{x\to 3} 3(x^2+9) = 3(9+9) = 54$$

Since $3(x^2+9)$ is continuous and equal to the first expression except at $x = 3$, the function may be made continuous by setting $Px + 9$ equal to 54. So,

$$Px + 9 = 54 \quad \text{and} \quad P = 15$$

(b) $g(x) = \begin{cases} \dfrac{x^3-27}{x^2-9}, & \text{if } x \neq 3 \\ P, & \text{if } x = 3 \end{cases}$

Factoring:

$$\lim_{x\to 3} \frac{x^3-27}{x^2-9} = \lim_{x\to 3} \frac{(x^2+3x+9)(x-3)}{(x+3)(x-3)} = \lim_{x\to 3} \frac{x^2+3x+9}{x+3} = \frac{9+9+9}{3+3} = \frac{27}{6} = \frac{9}{2}$$

Since $\dfrac{x^2+3x+9}{x+3}$ is continuous except at $x = -3$ and equal to $\dfrac{x^3-27}{x^2-9}$ except at $x = 3$, the function may be made continuous by letting P equal to $\frac{9}{2}$.

9. Determine the largest intervals on which the functions defined are continuous.

(a) $f(x) = \begin{cases} 8 - 7x, & \text{if } x \leq 4 \\ -x - 16, & \text{if } x > 4 \end{cases}$ (b) $f(x) = \begin{cases} x^2 + 1, & \text{if } x \leq -3 \\ 5 - x, & \text{if } x > -3 \end{cases}$

Solution:

(a) Since linear functions are always continuous, the only possibility for discontinuity is at $x = 4$:

$$\begin{aligned} \lim_{x\to 4^-} f(x) &= \lim_{x\to 4^-} 8 - 7x = 8 - 28 = -20 \\ \lim_{x\to 4^+} f(x) &= \lim_{x\to 4^+} -x - 16 = -20 \\ f(4) &= 8 - 28 = -20 \end{aligned}$$

Thus, the function is everywhere continuous, i.e. on $(-\infty, +\infty)$.

(b) Since linear and quadratic functions are continuous, the only possibility for discontinuity is at $x = -3$.

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2+1) = 3^2+1 = 10$$
$$f(3) = 3^2+1 = 10$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (5-x) = 5-3 = 2$$

So intervals $\lim_{x\to 3} f(x)$ does not exist.
The intervals are $(-\infty, -3)$ and $(-3, \infty)$

10. Identify all asymptotes of the graphs following.

(a) $y = \dfrac{x-2}{x-1}$

(b) $y = \dfrac{x^2-3x+2}{x^2-4}$

(c) $y = \dfrac{\sqrt{x^2+4}}{x}$

(d) $y = \dfrac{x^2-9}{x^2-5x+6}$

Solution:

(a) $y = \dfrac{x-2}{x-1}$

Horizontal:

$$\lim_{x\to\infty} \frac{x-2}{x-1} = 1$$
$$\lim_{x\to-\infty} \frac{x-2}{x-1} = 1$$

Horizontal asymptotes in both directions:

$$y = 1$$

Vertical asymptotes: where denominator is 0:

$$x - 1 = 0$$
$$x = 1$$

(b) $y = \dfrac{x^2-3x+2}{x^2-4}$

Horizontal asymptotes:

$$\lim_{x\to\infty} \frac{x^2 - 3x + 2}{x^2 - 4} = 1$$
$$\lim_{x\to-\infty} \frac{x^2 - 3x + 2}{x^2 - 4} = 1$$
$$y = 1$$

Vertical asymptotes: where denominator is 0

$$x + 2 = 0$$
$$x = -2$$

$(x - 2)$ does not yield an asymptote, since it results only in a point discontinuity.

(c) $y = \dfrac{\sqrt{x^2 + 4}}{x}$

Vertical: Denominator is 0 at $x = 0$.

Vertical asymptote: $x = 0$

Horizontal asymptotes:
For $x > 0$:

$$y = \frac{\sqrt{x^2 + 4}}{x} = \frac{\sqrt{x^2 + 4} \cdot \frac{1}{x}}{x \cdot \frac{1}{x}} = \frac{\sqrt{\frac{x^2+4}{x^2}}}{1} = \sqrt{1 + \frac{4}{x^2}}$$
$$\lim_{x\to\infty} y = \lim_{x\to\infty} \sqrt{1 + \frac{4}{x^2}} = 1$$

For $x < 0$:

$$y = \frac{\sqrt{x^2 + 4}}{x} = \frac{\sqrt{x^2 + 4} \cdot \frac{1}{x}}{x \cdot \frac{1}{x}} = \frac{-\sqrt{\frac{x^2+4}{x^2}}}{1} = -\sqrt{1 + \frac{4}{x^2}}$$
$$\lim_{x\to-\infty} y = \lim_{x\to-\infty} -\sqrt{1 + \frac{4}{x^2}} = -1$$

Asymptotes: $y = 1$, $y = -1$

(d) $y = \dfrac{x^2 - 9}{x^2 - 5x + 6} = \dfrac{(x + 3)(x - 3)}{(x - 2)(x - 3)} = \dfrac{x + 3}{x - 2}\ (x \neq 3)$

Vertical asymptotes: at $x = 2$
Horizontal asymptotes:
For $x \neq 3$,

$$\begin{aligned} y =& \frac{x+3}{x-2} = \frac{x-2}{x-2} + \frac{5}{x-2} = 1 + \frac{5}{x-2} \\ \lim_{x\to\infty} y =& \lim_{x\to\infty} 1 + \frac{5}{x-2} = 1 \\ \lim_{x\to-\infty} y =& \lim_{x\to-\infty} 1 + \frac{5}{x-2} = 1 \end{aligned}$$

$$x = 2,\ y = 1$$

11. Give a specific example to show that it is possible for $\lim_{x\to a} f(x)$ to exist when $f(a)$ is undefined.

Solution:

$$f(x) = \frac{x^2-1}{x-1}, a = 1.\ f(1) \text{ does not exist, but } \lim_{x\to 1} f(x) = 2.$$

Other Examples:

$$\begin{aligned} f(x) =& \frac{x^2+2x+1}{x+1},\ a = -1.\ f(-1) \text{ does not exist, but } \lim_{x\to-1} f(x) = 0. \\ f(x) =& \frac{\sin x}{x},\ a = 0.\ f(0) \text{ does not exist, but } \lim_{x\to 0} f(x) = 1. \end{aligned}$$

12. Let $y = \dfrac{7}{\sqrt{x^4 - 15}}$. Determine y' when $x = 2$.

Solution:

$$\begin{aligned} y =& \frac{7}{\sqrt{x^4-15}} \\ =& \frac{7}{(x^4-15)^{1/2}} \\ =& 7(x^4-15)^{-1/2} \end{aligned}$$

$$y' = 7 \cdot -\frac{1}{2} \cdot (x^4-15)^{-3/2} \cdot 4x^3$$

$$
\begin{aligned}
&= -\frac{7}{2} \cdot 4 \cdot (x^4 - 15)^{-3/2} \cdot x^3 \\
&= -14x^3(x^4 - 15)^{-3/2}
\end{aligned}
$$

So

$$
\begin{aligned}
y'(2) &= -14 \cdot 2^3(2^4 - 15)^{-3/2} \\
&= -14 \cdot 8 \cdot (16 - 15)^{-3/2} \\
&= -14 \cdot 8 \cdot 1 \\
&= -112
\end{aligned}
$$

13. Let $f(x) = \sqrt[3]{(2x + 17)^2}$. Determine $f'(5)$.

Solution:

$$
\begin{aligned}
f(x) &= \left[(2x + 17)^2\right]^{1/3} \\
&=(2x + 17)^{2 \cdot 1/3} \\
&=(2x + 17)^{2/3} \\
f'(x) &=\frac{2}{3}(2x + 17)^{-1/3} \cdot 2 \\
f'(5) &=\frac{2}{3}(10 + 17)^{-1/3} \cdot 2 \\
&=\frac{2}{3}(27)^{-1/3} \cdot 2 \\
&=\frac{2}{3} \cdot \frac{1}{3} \cdot 2 \\
&=\frac{4}{9}
\end{aligned}
$$

14. Determine y' when $x = 1$ if $y = \dfrac{8x}{3} - \dfrac{3}{8x}$.

Solution:

$$
\begin{aligned}
y &=\frac{8}{3}x - \frac{3}{8}x^{-1} \\
y' &=\frac{8}{3} + \frac{3}{8}x^{-2}
\end{aligned}
$$

$$
\begin{aligned}
y'(1) =& \frac{8}{3} + \frac{3}{8}(1)^{-2} \\
=& \frac{8}{3} + \frac{3}{8} \\
=& \frac{8 \cdot 8 + 3 \cdot 3}{3 \cdot 8} \\
=& \frac{64 + 9}{24} = \frac{73}{24}
\end{aligned}
$$

15. Determine y' when $x = 1$ if $y = (2 - x)\sqrt{x^2 + 8}$.

    Solution:

$$
\begin{aligned}
y =& (2 - x)(x^2 + 8)^{1/2} \\
y' =& (2 - x) \cdot \frac{1}{2}(x^2 + 8)^{-1/2} \cdot 2x + (-1)(x^2 + 8)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
y'(1) =& (2 - 1) \cdot \frac{1}{2}(1^2 + 8)^{-1/2} \cdot 2 \cdot 1 + (-1)(1^2 + 8)^{1/2} \\
=& 1 \cdot \frac{1}{2}(9)^{-1/2} \cdot 2 - (9)^{1/2} \\
=& 9^{-1/2} - 9^{1/2} \\
=& \frac{1}{3} - 3 = \frac{1}{3} - \frac{9}{3} = -\frac{8}{3}
\end{aligned}
$$

16. Find an equation of the tangent line to the curve defined by $y = 2x^2 - 5x + 8$ when $x = 2$.

    Solution:
    Find the first derivative:

$$y' = 4x - 5$$

The slope of the tangent curve is given by $y'(2)$:

$$m = y'(2) = 4 \cdot 2 - 5 = 8 - 5 = 3.$$

We also need a point on the line:
(Use the point where the tangent line intersects the curve.)

$$
\begin{aligned}
y =& 2 \cdot 2^2 - 5 \cdot 2 + 8 \\
=& 8 - 10 + 8 = 16 - 10 = 6
\end{aligned}
$$

So (2, 6) is on the line. The equation has the form

$$\begin{aligned}(y-6) &= 3(x-2)\\ y-6 &= 3(x-2)\\ y &= 3(x-2)+6\\ y &= 3x-6+6\\ y &= 3x\end{aligned}$$

17. At what point (x, y) is the tangent line to the curve $y = 2x^2 - 5x + 8$ parallel to the line $y = 3x - 7$

    Solution:

    - Slope of the given line is 3.
    - Need to find x such that $y' = 3$.

$$\frac{dy}{dx} = 4x - 5$$

$$\begin{aligned}\text{Let } 4x-5 &= 3\\ 4x &= 8\\ x &= 2\end{aligned}$$

$$\begin{aligned}\text{Find } y(2)\text{: } y(2) &= 2\cdot 2^2 - 5\cdot 2 + 8\\ &= 8 - 10 + 8 = 6\end{aligned}$$

At $(x, y) = (2, 6)$, the tangent line is parallel to $y = 3x - 7$.

18. Find the slope of the line tangent to the graph of $x^2 + 2xy^2 + 3y = 31$ at the point $(2, -3)$.

    Solution:

    A. Verify that $(2, -3)$ is on the graph:

$$\begin{aligned}(2)^2 + 2(2)(-3)^2 + 3(-3) &= 31\\ 4 + 4\cdot 9 + (-9) &= 31\\ 4 + 36 - 9 &= 31\\ 40 - 9 &= 31\checkmark\end{aligned}$$

B. Differentiate Implicitly

$$\frac{d}{dx}(x^2 + 2xy^2 + 3y) = \frac{d}{dx}(31)$$

$$2x + 2x2yy' + 2y^2 + 3y' = 0$$

C. Let $x = 2$ and $y = -3$.

$$\begin{aligned} 2(2) + 2(2)(2)(-3)y' + 2(-3)^2 + 3y' =& 0 \\ 4 + -24y' + 2 \cdot 9 + 3y' =& 0 \\ 4 + 18 - 24y' + 3y' =& 0 \\ 22 - 21y' =& 0 \\ -21y' =& -22 \\ y' =& \frac{22}{21} \leftarrow \text{slope of the desired tangent line} \end{aligned}$$

To find the tangent line itself:

$$\begin{aligned} (y - (-3)) =& \frac{22}{21}(x - 2) \\ y + 3 =& \frac{22}{21}(x - 2) \end{aligned}$$

19. Let $y = \sqrt{x^2 - 1}$. Find y'' when $x = 2$.

Solution:

$$\begin{aligned} y =& (x^2 - 1)^{1/2} \\ y' =& \frac{1}{2}(x^2 - 1)^{-1/2} \cdot 2x \\ y' =& (x^2 - 1)^{-1/2}x \\ y'' =& -\frac{1}{2}(x^2 - 1)^{-3/2}2x \cdot x + (x^2 - 1)^{-1/2} \cdot 1 \\ =& -(x^2 - 1)^{-3/2}x^2 + (x^2 - 1)^{-1/2} \\ =& -(x^2 - 1)^{-1} \cdot (x^2 - 1)^{-1/2} \cdot x^2 + (x^2 - 1)^{-1/2} \\ =& \left[-(x^2 - 1)^{-1} \cdot x^2 + 1\right](x^2 - 1)^{-1/2} \\ =& \left[-\frac{x^2}{x^2 - 1} + 1\right](x^2 - 1)^{-1/2} \end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{-x^2}{x^2-1} + \frac{x^2-1}{x^2-1} \right] (x^2-1)^{-1/2} \\
&= \frac{-1}{x^2-1} (x^2-1)^{-1/2} \\
y''(2) &= \frac{-1}{4-1} (2^2-1)^{-1/2} = -\frac{1}{3} (3)^{-1/2} \\
&= -\frac{1}{3\sqrt{3}} = -\frac{\sqrt{3}}{9}
\end{aligned}
$$

20. Let $f(x) = \sqrt{x + \sqrt{x}}$. Find $f'(1)$.

Solution:

$$
\begin{aligned}
f(x) &= \left(x + x^{1/2}\right)^{1/2} \\
f'(x) &= \frac{1}{2} \left(x + x^{1/2}\right)^{-1/2} \left[1 + \frac{1}{2} x^{-1/2}\right] \\
f'(1) &= \frac{1}{2} \left(1 + 1^{1/2}\right)^{-1/2} \left[1 + \frac{1}{2} \cdot 1^{-1/2}\right] \\
f'(1) &= \frac{1}{2} \cdot 2^{-1/2} \left[\frac{3}{2}\right] \\
f'(1) &= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \cdot \frac{3}{2} = \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{3}{2} = \frac{3\sqrt{2}}{8}
\end{aligned}
$$

21. Determine $f'(1)$ if $f(x) = \sqrt[3]{\dfrac{x}{x^3+1}}$.

Solution:
Recall $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - g'f}{g^2}$.

$$
\begin{aligned}
f(x) &= \left(\frac{x}{x^3+1}\right)^{1/3} \\
f'(x) &= \frac{1}{3} \left(\frac{x}{x^3+1}\right)^{-2/3} \cdot \frac{1 \cdot (x^3+1) - (x^3+1)' \cdot x}{(x^3+1)^2} \\
&= \frac{1}{3} \left(\frac{x}{x^3+1}\right)^{-2/3} \cdot \frac{(x^3+1) - 3x^2 \cdot x}{(x^3+1)^2}
\end{aligned}
$$

$$
\begin{aligned}
f'(1) =& \frac{1}{3}\left(\frac{1}{1^3+1}\right)^{-2/3} \cdot \frac{(1^3+1) - 3\cdot 1^2 \cdot 1}{(1^3+1)^2} \\
=& \frac{1}{3}\left(\frac{1}{2}\right)^{-2/3} \cdot \frac{2-3}{2^2} \\
=& \frac{1}{3}\left(\frac{1}{2}\right)^{-2/3} \cdot \frac{-1}{4} \\
=& \frac{-1}{12}(4)^{1/3} = -\frac{\sqrt[3]{4}}{12}
\end{aligned}
$$

22. Determine $f'(3)$ if $f(x) = \dfrac{1}{x - \sqrt{x^2-5}}$.

Solution:

$$f(x) = \left[x - (x^2-5)^{1/2}\right]^{-1}$$

$$
\begin{aligned}
f'(x) &= -1 \cdot \left[x - (x^2-5)^{1/2}\right]^{-2}\left[1 - \frac{1}{2}(x^2-5)^{-1/2} \cdot 2x\right] \\
f'(3) &= -1 \cdot \left[3 - (9-5)^{1/2}\right]^{-2}\left[1 - \frac{1}{2}(9-5)^{-1/2} \cdot 2 \cdot 3\right] \\
f'(3) &= -1 \cdot \left[3 - 4^{1/2}\right]^{-2}\left[1 - \frac{1}{2}(4)^{-1/2} \cdot 2 \cdot 3\right] \\
f'(3) &= -1 \cdot [3-2]^{-2}\left[1 - \frac{1}{2} \cdot \frac{1}{2} \cdot 2 \cdot 3\right] \\
f'(3) &= -1 \cdot [1]^{-2}\left[1 - \frac{3}{2}\right] \\
&= -1 \cdot 1 \cdot \left[-\frac{1}{2}\right] = +\frac{1}{2}
\end{aligned}
$$

23. Determine y'' at $x = 1$ if $y = 3\sqrt[3]{x^4} - \dfrac{1}{3x^3}$.

Solution:

$$
\begin{aligned}
y =& 3x^{4/3} - \frac{1}{3}x^{-3} \\
y' =& 3 \cdot \frac{4}{3}x^{1/3} - \frac{1}{3}(-3)x^{-4} \\
y' =& 4x^{1/3} + x^{-4}
\end{aligned}
$$

$$
\begin{aligned}
y'' &= \frac{4}{3}x^{-2/3} - 4x^{-5} \\
y''(1) &= \frac{4}{3}(1)^{-2/3} - 4(1)^{-5} \\
&= \frac{4}{3} - 4 = \frac{4}{3} - \frac{12}{3} = -\frac{8}{3}
\end{aligned}
$$

24. Let $y = [\cos(2x - \pi)]^3$. Find y' at $x = \frac{\pi}{6}$.

    Solution:

$$
\begin{aligned}
y &= [\cos(2x - \pi)]^3 \\
y' &= 3\,[\cos(2x - \pi)]^2 \cdot [-\sin(2x - \pi)] \cdot 2 \\
y'\left(\frac{\pi}{6}\right) &= 3\left[\cos\left(2 \cdot \frac{\pi}{6} - \pi\right)\right]^2 \cdot \left[-\sin\left(2 \cdot \frac{\pi}{6} - \pi\right)\right] \cdot 2 \\
&= 3\left[\cos\left(\frac{\pi}{3} - \pi\right)\right]^2 \cdot \left[-\sin\left(\frac{\pi}{3} - \pi\right)\right] \cdot 2 \\
&= 3\left[\cos\left(\frac{-2\pi}{3}\right)\right]^2 \cdot \left[-\sin\left(\frac{-2\pi}{3}\right)\right] \cdot 2 \\
&= 6 \cdot \left[-\frac{1}{2}\right]^2 \cdot \left[-\left(-\frac{\sqrt{3}}{2}\right)\right] \\
&= 6 \cdot \frac{1}{4} \cdot \frac{\sqrt{3}}{2} \\
&= \frac{6}{8}\sqrt{3} = \frac{3}{4}\sqrt{3}
\end{aligned}
$$

25. Determine $f'(\frac{\pi}{4})$ if $f(x) = \dfrac{\tan x}{1 + \cos x}$.

    Solution:
    Use Quotient Rule:

$$
\begin{aligned}
f'(x) &= \frac{(\sec x)^2\,[1 + \cos x] - \tan x\,[-\sin x]}{[1 + \cos x]^2} \\
f'(x) &= \frac{\left(\sec \frac{\pi}{4}\right)^2 \left[1 + \cos \frac{\pi}{4}\right] - \tan \frac{\pi}{4} \left[-\sin \frac{\pi}{4}\right]}{\left[1 + \cos \frac{\pi}{4}\right]^2}
\end{aligned}
$$

Recall:

$$\begin{aligned}\cos\frac{\pi}{4} &= \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}\\ \tan\frac{\pi}{4} &= 1\\ \sec\frac{\pi}{4} &= \sqrt{2}\\ \left(\sec\frac{\pi}{4}\right)^2 &= 2\end{aligned}$$

$$\begin{aligned}f'\left(\frac{\pi}{4}\right) &= \frac{2\cdot\left[1+\frac{\sqrt{2}}{2}\right]-1\cdot\left[-\frac{\sqrt{2}}{2}\right]}{\left[1+\frac{\sqrt{2}}{2}\right]^2}\\ &= \frac{2+\sqrt{2}+\frac{\sqrt{2}}{2}}{1+2\left(\frac{\sqrt{2}}{2}\right)+\left[\frac{\sqrt{2}}{2}\right]^2}\\ &= \frac{2+\sqrt{2}+\frac{\sqrt{2}}{2}}{1+\sqrt{2}+\frac{1}{2}}\end{aligned}$$

Noticing similarity between numerator and denominator:

$$\begin{aligned}f'\left(\frac{\pi}{4}\right) &= \frac{\left(2+\sqrt{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}}{\left(1+\sqrt{2}+\frac{1}{2}\right)\sqrt{2}}\\ &= \frac{\left(2+\sqrt{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}}{\left(\sqrt{2}+2+\frac{\sqrt{2}}{2}\right)} = \sqrt{2}\end{aligned}$$

Alternatively, to do the simplification we could write:

$$\begin{aligned}f'(x) &= \frac{2+\sqrt{2}+\frac{\sqrt{2}}{2}}{1+\sqrt{2}+\frac{1}{2}}\\ &= \frac{2+\frac{3}{2}\sqrt{2}}{\frac{3}{2}+\sqrt{2}}\cdot\frac{\frac{3}{2}-\sqrt{2}}{\frac{3}{2}-\sqrt{2}}\end{aligned}$$

$$
\begin{aligned}
&= \frac{3 + \left(\frac{9}{4} - 2\right)\sqrt{2} - \frac{3}{2}\cdot 2}{\frac{9}{4} - 2} \\
&= \frac{\frac{1}{4}\sqrt{2}}{\frac{1}{4}} = \sqrt{2}
\end{aligned}
$$

26. Let $f(x) = \sin 3x \cos 2x$. Find $f'\left(\frac{\pi}{6}\right)$.

    Solution:
    Use Product Rule

$$
\begin{aligned}
f'(x) &= [\sin 3x \cos 2x]' \\
&= [\sin 3x]' (\cos 2x) + (\sin 3x)[\cos 2x]' \\
&= 3\cos 3x \cdot \cos 2x + \sin 3x \cdot (-2\sin 2x)
\end{aligned}
$$

$$
\begin{aligned}
f'\left(\frac{\pi}{6}\right) &= 3\cos\left(3\cdot\frac{\pi}{6}\right)\cdot\cos\left(2\cdot\frac{\pi}{6}\right) + \sin\left(3\cdot\frac{\pi}{6}\right)\cdot\left(-2\sin 2\cdot\frac{\pi}{6}\right) \\
&= 3\cos\left(\frac{\pi}{2}\right)\cdot\cos\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{2}\right)\cdot\left(-2\sin\frac{\pi}{3}\right) \\
&= 3\cdot 0\cdot\cos\left(\frac{\pi}{3}\right) + 1\cdot\left(-2\cdot\frac{\sqrt{3}}{2}\right) \\
&= 0 + \left(-\sqrt{3}\right) = -\sqrt{3}
\end{aligned}
$$

$$
\begin{aligned}
\cos\frac{\pi}{2} &= \frac{0}{1} = 0 \\
\sin\frac{\pi}{2} &= \frac{1}{1} = 1 \\
\sin\frac{\pi}{3} &= \frac{\sqrt{3}}{2}
\end{aligned}
$$

27. Let $y^4 + x^4 - 2x^2y + 9x = 9$. Find y' at $(1, 1)$.

    Solution:
    Use implicit differentiation, but group terms involving y together first. (This isn't all that important, but it makes it less confusing to do it before the equation gets messier.)

$$
y^4 - 2x^2y + x^4 + 9x = 9
$$

$$\begin{aligned} \frac{d}{dx}(y^4 - 2x^2y + x^4 + 9x) =& \frac{d}{dx}(9) \\ 4y^3y' - 2x^2y' - 4xy + 4x^3 + 9 =& 0 \end{aligned}$$

Plug in $y = 1$ and $x = 1$:

$$\begin{aligned} 4 \cdot 1^3 \cdot y' - 2 \cdot 1^2 \cdot y' - 4 \cdot 1 \cdot 1 + 4 \cdot 1^3 + 9 =& 0 \\ 4y' - 2y' - 4 + 4 + 9 =& 0 \\ 2y' + 9 =& 0 \\ 2y' =& -9 \\ y' =& -\frac{9}{2} \end{aligned}$$

28. Suppose that $f'(x) = x^2(4x - 3)$ and $f''(x) = 6x(2x - 1)$. Determine the intervals on which f is increasing.

Solution:

Use sign analysis (or another method if you prefer):

$$\begin{aligned} \text{Need:} \qquad f'(x) >& 0 \\ x^2(4x - 3) >& 0 \\ x^2 \cdot \left(x - \frac{3}{4}\right) \cdot 4 >& 0 \end{aligned}$$

x^2	$++++$	$++++$	$++++$
$x - \frac{3}{4}$	$----$	$----$	$++++$
$x^2(4x - 3)$	$----$	$----$	$++++$
		0	$\frac{3}{4}$

$x^2(4x - 3) > 0$ exactly when $x > \frac{3}{4}$.
So $f'(x) > 0$ on $(\frac{3}{4}, +\infty)$.
So f is increasing on $\left[\frac{3}{4}, +\infty\right)$.

29. Suppose that $f'(x) = \dfrac{x - 1}{2x + 3}$ and $f''(x) = \dfrac{5}{(2x + 3)^2}$. Determine the intervals on which f is decreasing.

Solution:

f is decreasing whenever $f'(x) < 0$:
Top, $x - 1$ is > 0 when

	$x<-\frac{3}{2}$	$-\frac{3}{2}<x<1$	$x>1$
$x-1$	$----$	$----$	$++++$
$2x+3$	$----$	$++++$	$++++$
$f'(x)$	$++++$	$----$	$++++$

f is decreasing on $\left(-\frac{3}{2}, 1\right]$
Note that $-\frac{3}{2}$ is excluded because $f'(-\frac{3}{2})$ is undefined and $f' \to -\infty$ as $x \to -\frac{3}{2}$ from above.

30. Determine the intervals on which f is concave upward if

$$f(x) = \frac{1}{10}x^5 + \frac{1}{6}x^4 - 4x^3 + 87x + 69.$$

Solution:
Differentiating:

$$\begin{aligned} f'(x) =& \frac{1}{10} \cdot 5x^4 + \frac{1}{6} \cdot 4x^3 - 4 \cdot 3x^2 + 87 \\ =& \frac{1}{2}x^4 + \frac{2}{3}x^3 - 12x^2 + 87 \\ f''(x) =& \frac{1}{2} \cdot 4x^3 + \frac{2}{3} \cdot 3x^2 - 12 \cdot 2x \\ =& 2x^3 + 2x^2 - 24x \end{aligned}$$

Factor $f''(x)$, if possible:

$$\begin{aligned} f''(x) =& 2(x^3 + x^2 - 12x) \\ =& 2x(x^2 + x - 12) \\ =& 2x(x+4)(x-3) \end{aligned}$$

Do sign analysis or otherwise determine where $f''(x)$ is positive:

	$(-\infty,-4)$	$(-4,0)$	$(0,3)$	$(3,\infty)$
x	− − − −	− − − −	+ + + +	+ + + +
$x+4$	− − − −	+ + + +	+ + + +	+ + + +
$x-3$	− − − −	− − − −	− − − −	+ + + +
$2x(x+4)(x-3)$	− − − −	+ + + +	− − − −	+ + + +

From sign analysis, $f''(x) > 0$ for $-4 < x < 0$ and $x > 3$
This gives as intervals for concave up:

$$(-4, 0) \text{ and } (3, \infty)$$

31. Let $f'(x) = \dfrac{-3}{2(x^2-4)^2}$ and $f''(x) = \dfrac{6x}{(x^2-4)^3}$. Determine the intervals on which f is concave downward.

Solution:

$$f'(x) = \frac{-3}{2(x^2-2)^2} = -\frac{3}{2}(x^2-2)^{-2}$$
$$f''(x) = -\frac{3}{2}(-2)(x^2-2)^{-3}(2x) = 6x(x^2-4)^3 = \frac{6x}{(x^2-4)^3}$$

f is concave downward when $f''(x) < 0$.

$$\frac{6x}{(x^2-4)^3} = \frac{6x}{(x+2)^3(x-2)^3} \quad \text{Zeros at: } 0, -2, +2$$

	$(-\infty,-2)$	$(-2,0)$	$(0,2)$	$(2,\infty)$
$6x$	− − − −	− − − −	+ + + +	+ + + +
$x+2$	− − − −	+ + + +	+ + + +	+ + + +
$x-2$	− − − −	− − − −	− − − −	+ + + +
$f''(x)$	− − − −	+ + + +	− − − −	+ + + +

f is concave downward on $(-\infty, -2)$ and $(0, 2)$.

32. Determine all points of inflection for $f(x) = \frac{1}{12}x^4 + \frac{1}{6}x^3 - 6x^2$.

    Solution:

$$f'(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 12x$$
$$f''(x) = x^2 + x - 12$$

$$f''(x) = 0$$
$$x^2 + x - 12 = 0$$
$$(x+4)(x-3) = 0$$
$$x = -4, +3$$

Since $x^2 + x - 12$ is a quadratic having two zeros, it must change sign at both zeros. hence points of inflection occur when $x = 4$ and $x = -3$.

33. Determine all points of inflection for $f(x) = \frac{1}{12}x^4 - \frac{7}{3}x^3 + \frac{49}{2}x^2 + 88$.

    Solution:

$$f'(x) = \frac{1}{3}x^3 - 7x^2 + 49x$$
$$f''(x) = x^2 - 14x + 49$$

$$f''(x) = 0$$
$$x^2 - 14x + 49 = 0$$
$$(x-7)(x-7) = 0$$
$$x = 7 \text{ is the only root.}$$

It is necessary to see if $f''(x)$ actually changes sign at $x = 7$, but it doesn't since $f''(x) = x^2 - 14x + 49 = (x-7)^2$ is a perfect square and cannot be negative.
No inflection points

34. Let $f(x) = \frac{1}{3}x^3 + 3x^2 + x - 2$. Determine all local minima for f.

    Solution:

$f'(x) = x^2 + 6x + 1$ $\qquad$ $f''(x) = 2x + 6$

$$x^2 + 6x + 1 = 0$$

$$\begin{aligned} x =& \frac{-6 \pm \sqrt{36 - 4 \cdot 1 \cdot 1}}{2} \\ =& -3 \pm \frac{\sqrt{32}}{2} \\ =& -3 \pm \frac{4\sqrt{2}}{2} \\ =& -3 \pm 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} f''(-3 + 2\sqrt{2}) =& 2(-3 + 2\sqrt{2}) + 6 \\ =& -6 + 4\sqrt{2} + 6 \\ =& 4\sqrt{2} > 0 \\ & \text{minimum} \\ f''(-3 - 2\sqrt{2}) =& 2(-3 - 2\sqrt{2}) + 6 \\ =& -6 - 4\sqrt{2} + 6 \\ =& -4\sqrt{2} < 0 \\ & \text{maximum} \end{aligned}$$

Minimum at $-3 + 2\sqrt{2} = 2\sqrt{2} - 3$.

35. Let $f(x) = \frac{1}{4}x^4 - \frac{5}{2}x^2 + 3$. Determine all local maxima for f.

Solution:

$$\begin{aligned} f'(x) =& x^3 - 5x \\ f'(x) =& 0 \\ x^3 - 5x =& 0 \\ x(x^2 - 5) =& 0 \\ x =& 0, \pm\sqrt{5} \\ f''(x) =& 3x^2 - 5 \\ f''(0) =& 3 \cdot 0^2 - 5 = -5 \text{ maximum} \end{aligned}$$

$$f''(+\sqrt{5}) = 3(\sqrt{5})^2 - 5 = 15 - 5 = 10 > 0 \text{ minimum}$$
$$f''(-\sqrt{5}) = 3(-\sqrt{5})^2 - 5 = 15 - 5 = 10 > 0 \text{ minimum}$$

Local maximum at 0

36. Find the maximum and minimum values of the functions on the given intervals.

(a) $f(x) = x^3 - 12x$ on the interval $[0, 3]$.

(b) $f(x) = x^3 - 12x$ on the interval $[-3, 0]$.

Solution:

(a)

$$f'(x) = 3x^2 - 12$$
$$f''(x) = 6x$$

Interval $[0, 3]$: Let $f'(x) = 0$

$$3x^2 - 12 = 0$$
$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = +2 \text{ or } -2$$

2 lies in the interval, -2 doesn't.
So check 0, 3, and 2:

$$f(0) = 0^3 - 12 \cdot 0 = 0$$
$$f(3) = 3^3 - 12 \cdot 3 = 27 - 36 = -9$$
$$f(2) = 2^3 - 12 \cdot 2 = 8 - 24 = -16$$

Minimum is -16, maximum is 0.

(b) Interval $[-3, 0]$: Let $f'(x) = 0$

$$3x^2 - 12 = 0$$
$$x^2 - 4 = 0$$
$$x^2 = 4$$
$$x = +2 \text{ or } -2$$

-2 lies in the interval, 2 doesn't.
So check -3, 0, and -2:

$$f(-3) = (-3)^3 - 12(-3) = -27 + 36 = 9$$
$$f(0) = 0^3 - 12 \cdot 0 = 0$$
$$f(-2) = (-2)^3 - 12(-2) = -8 + 24 = 16$$

Minimum is 0, maximum is 16.

37. A rock thrown from the top of a cliff is $s(t) = 192 + 64t - 16t^2$ feet above the ground t seconds after being thrown. Determine

   (a) the height of the cliff,

   (b) the time it takes the rock to reach the ground, and

   (c) the velocity of the rock when it strikes the ground.

   Solution:

   (a) The height of the cliff is the same as the height of the rock immediately after being thrown.
   This is given by $s(0) = 192 + 64 \cdot 0 - 16 \cdot 0^2 = 192$.
   The cliff is 192 feet high.

   (b) t to reach the ground is given by solving for t in $s(t) = 0$, i.e. solving for the rock being zero feet above the ground.

$$s(t) = 0$$
$$0 = 192 + 64t - 16t^2$$

   Note: $192 = 12 \cdot 16$ and $64 = 4 \cdot 16$, so

$$0 = 16(12 + 4t - t^2)$$
$$t^2 - 4t - 12 = 0$$
$$(t - 6)(t + 2) = 0$$

$$t = 6 \text{ or } t = -2$$

   $t = -2$ is not appropriate to question. So, it takes 6 seconds for the rock to reach the ground.

(c) To find the velocity of the rock when it strikes the ground, find $s'(6)$:

$$\begin{aligned} s(t) =& 192 + 64t - 16t^2 \\ s'(t) =& 0 + 64 - 2 \cdot 16t = 64 - 32t \\ s'(6) =& 64 - 32 \cdot 6 = 64 - 192 = -128 \end{aligned}$$

The rock is traveling at 128 feet per second when it strikes the ground. [The minus sign indicates that the direction of the motion is downward, because the expression for $s(t)$ gives altitude above the ground.]

38. A rock is thrown vertically upward from the roof of a house 32 feet high with an initial velocity of 128 ft/sec.

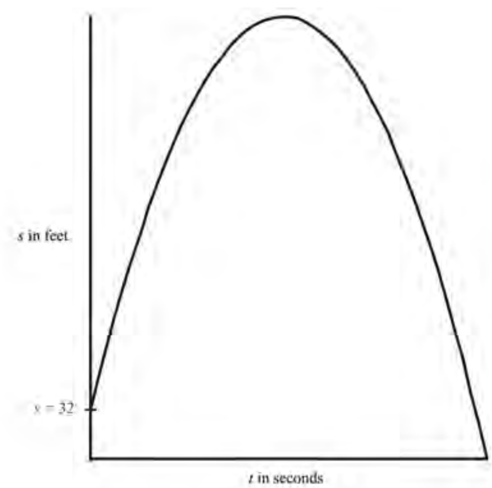
(a) What is the speed of the rock at the end of 2 seconds?

(b) What is the maximum height the rock will reach?

Solution:

(a) Leaving off units:

$$\begin{aligned} a =& -32 \\ v_0 =& +128 \\ s_0 =& +32 \\ a = \frac{dv}{dt}&, v = \frac{ds}{dt} \end{aligned}$$



So,

$$\begin{aligned} v =& \int a(t)\, dt + c_1 \\ v =& \int -32\, dt + c_1 \\ =& -32t + c_1 \\ v(0) =& +128 \end{aligned}$$

$$
\begin{aligned}
v(0) &= -32 \cdot 0 + c_1 \\
c_1 &= +128 \\
v(t) &= 128 - 32t \\
s(t) &= \int v(t)\,dt + c_2 \\
&= \int (128 - 32t)\,dt + c_2 \\
&= 128t - \frac{32}{2}t^2 + c_2 \\
32 = s_0 &= 128 \cdot 0 - \frac{32}{2} \cdot 0^2 + c_2 \\
32 &= c_2
\end{aligned}
$$

So,

$$
\begin{aligned}
s(t) &= 128t - \frac{32}{2}t^2 + 32 \\
s(t) &= 128t - 16t^2 + 32 \\
v(t) &= 128 - 32t
\end{aligned}
$$

Recall from previous work: $a(t) = -32$

To find speed after 2 seconds, first check s to see that rock has not hit the ground:

$$
\begin{aligned}
s(2) &= 128 \cdot 2 - 16 \cdot 2^2 + 32 \\
&= 256 - 64 + 32 \\
&= 256 - 32 = 224 > 0 \\
v(2) &= 128 - 32 \cdot 2 = 128 - 64 = 64
\end{aligned}
$$

Velocity after 2 seconds is 64 feet/sec. upward.

Remark: Because $s(t) = -16t^2 + 128t + 32$ has form of parabola, opening downwards, and the initial velocity is directed upward, we know that once $s(t)$ reaches 0, it will continue decreasing. Thus, $s(2) > 0$ guarantees that the rock does not hit the ground before or at 2 seconds. Note that once the rock hits the ground, none of the equations apply.

Alternatively: Since $v(2) > 0$, we know that the maximum height has not been reached, so the rock cannot have hit the ground.

(b) To find maximum height:
Maximize $s(t)$:

$$
s'(t) = v(t) = 128 - 32t
$$

$$\begin{aligned} \text{Let } v(t) &= 0 \\ 128 - 32t &= 0 \\ 128 &= 32t \\ 32t &= 128 \\ t &= \frac{128}{32} = 4 \end{aligned}$$

Maximum height is at 4 seconds.

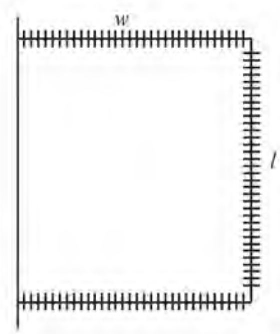
$$\begin{aligned} s(t) &= -16t^2 + 128t + 32 \\ \text{max height } = s(4) &= -16 \cdot 4^2 + 128 \cdot 4 + 32 \\ &= -16 \cdot 16 + 128 \cdot 4 + 32 \\ &= -256 + 512 + 32 \\ &= 256 + 32 = 288 \end{aligned}$$

Maximum height is 288 feet.

39. What is the maximum area which can be enclosed by 200 ft. of fencing if the enclosure is in the shape of a rectangle and one side of the rectangle requires no fencing?

Solution:

Let w be the length of one of the sides whose opposite requires fencing and l the length of the side whose opposite does not.



Then we want to maximize $A = l \cdot w$ where $l + 2w = 200$.

$$\begin{aligned} l &= 200 - 2w \\ A = l \cdot w &= (200 - 2w)w = 200w - 2w^2 \end{aligned}$$

To maximize A, find the derivative and set it equal to 0:

$$\frac{dA}{dw} = 200 - 4w = 0$$

$$\begin{aligned}200 =& 4w\\ 4w =& 200\\ w =& 50\end{aligned}$$

Plug in to find maximum area:

$$\begin{aligned}A_{\max} =& \left[200w - 2w^2\right]_{w=50}\\ =& \left[200 \cdot 50 - 2 \cdot 50^2\right]\\ =& 10000 - 5000 = 5000\end{aligned}$$

The maximum area that can be enclosed is 5000 sq. feet.

40. A woman throws a ball vertically upward from the ground. The equation of its motion is given by $s(t) = -16t^2 + ct$, where c is the initial velocity of the ball. She wants the ball to reach a maximum height of 100 ft. Find c.

    Solution:

    To find maximum height given c, differentiate $s(t)$ and set the derivative equal to 0.

$$\begin{aligned}s'(t) = -32t + c =& 0\\ -32t =& -c\\ t =& \frac{-c}{-32} = \frac{c}{32}\end{aligned}$$

Find $s\left(\frac{c}{32}\right)$ and set it equal to 100.

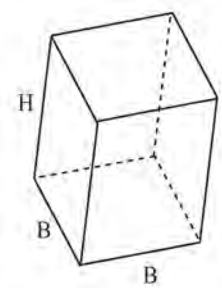
$$\begin{aligned}s\left(\frac{c}{32}\right) =& 100\\ -16\left(\frac{c}{32}\right)^2 + c\left(\frac{c}{32}\right) =& 100\\ -16\frac{c^2}{32^2} + \frac{c^2}{32} =& 100\\ \frac{-c^2}{64} + \frac{c^2}{32} =& 100\\ \frac{-c^2}{64} + \frac{2c^2}{64} =& 100\\ \frac{c^2}{64} =& 100\end{aligned}$$

$$
\begin{aligned}
c^2 =& 64 \cdot 100 = 6400 \\
c =& \sqrt{6400} = 80 \\
c =& 80
\end{aligned}
$$

41. A rectangular open tank is to have a square base, and its volume is to be 125 yd^3. The cost per square yard for the base is \$8 and for the sides is \$4. Find the dimensions of the tank in order to minimize the cost of the material.

Solution:

Let B be the length of the side of the base and H be the height (in yards).
Volume is given by $V = B \cdot B \cdot H = B^2 H$
Requirement: $V = 125$, $B^2 \cdot H = 125$



Also, must minimize total cost:

$$
\begin{aligned}
C =& 8 \cdot B^2 + 4 \cdot (4 \cdot BH) \\
C =& 8B^2 + 16BH \\
H =& \frac{125}{B^2} \\
C =& 8B^2 + 16B\frac{125}{B^2} \\
C =& 8B^2 + \frac{16 \cdot 125}{B} = 8B^2 + 16 \cdot 125B^{-1}
\end{aligned}
$$

Minimize total cost: $\dfrac{dC}{dB} = 16B + (-1)16 \cdot 125B^{-2}$

$$
\begin{aligned}
\frac{dC}{dB} = 16B - 16 \cdot 125B^{-2} =& 0 \\
B - 125B^{-2} =& 0 \\
B =& 125B^{-2} \\
B^3 =& 125
\end{aligned}
$$

$$B = 5$$

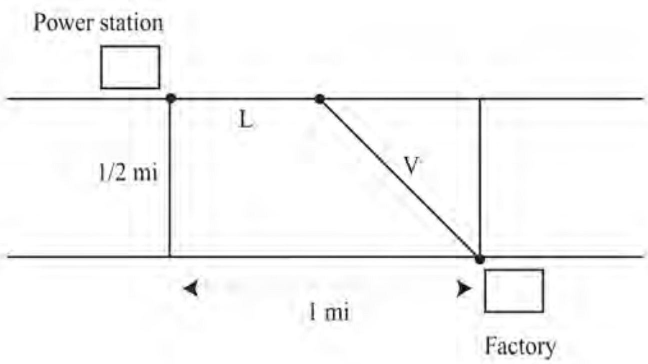
Total cost is minimized when $B = 5$.

So, $H = \frac{125}{B^2} = \frac{125}{25} = 5$

So, make a tank 5yd × 5yd × 5yd to minimize total cost.

42. A power station is on one side of a river which is 1/2 mile wide, and a factory is 1 mile downstream on the other side of the river. It costs $300 per foot to run power lines overland and $500 per foot to run them under water. Find the most economical way to run the power lines from the power station to the factory.

Solution:



Let V be the length under water, and L the length above water, in miles.

$$\text{Thus: } V^2 = (1 - L)^2 + \left(\frac{1}{2}\right)^2$$

Also, let C be total cost, and α be the number of feet in a mile. Thus:

$$C = 300\alpha L + 500\alpha V$$

Our goal is to maximize C with respect to L (and V).

$$\text{We know } C = 300\alpha L + 500\alpha V$$

$$\text{and also that: } V^2 = (1 - L)^2 + \left(\frac{1}{2}\right)^2$$

$$V = \sqrt{(1 - L)^2 + \left(\frac{1}{2}\right)^2}$$

$$\text{so that: } C = 300\alpha L + 500\alpha\sqrt{(1-L)^2 + \left(\frac{1}{2}\right)^2}$$

and differentiating:

$$\begin{aligned}
\frac{dC}{dL} &= 300\alpha + 500\alpha\frac{1}{2}\frac{1}{\sqrt{(1-L)^2 + \left(\frac{1}{2}\right)^2}}2(1-L)(-1)\\
&= 300\alpha + 500\alpha\frac{(-1)(1-L)}{\sqrt{(1-L)^2 + \left(\frac{1}{2}\right)^2}}\\
&= 300\alpha - 500\alpha\frac{(1-L)}{\sqrt{(1-L)^2 + \left(\frac{1}{2}\right)^2}}
\end{aligned}$$

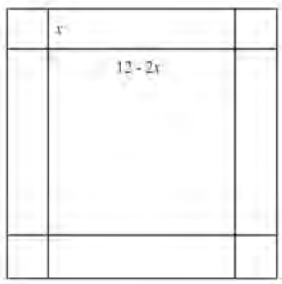
$$\begin{aligned}
\frac{dC}{dL} &= 0\\
300\alpha - 500\alpha\frac{(1-L)}{\sqrt{(1-L)^2 + \left(\frac{1}{2}\right)^2}} &= 0\\
500\alpha\frac{(1-L)}{\sqrt{(1-L)^2 + \left(\frac{1}{2}\right)^2}} &= 300\alpha\\
5\frac{(1-L)}{\sqrt{(1-L)^2 + \left(\frac{1}{2}\right)^2}} &= 3\\
5(1-L) &= 3\sqrt{(1-L)^2 + \left(\frac{1}{2}\right)^2}\\
25(1-L)^2 &= 9\left[(1-L)^2 + \left(\frac{1}{2}\right)^2\right]\\
25(1-L)^2 &= 9(1-L)^2 + \frac{9}{4}\\
16(1-L)^2 &= \frac{9}{4}\\
(1-L)^2 &= \frac{9}{64}\\
1-L &= \sqrt{\frac{9}{64}} = \frac{3}{8}\\
1-L &= \frac{3}{8}
\end{aligned}$$

$$L = \frac{5}{8}$$

The power line should run $\frac{5}{8}$ mile overland.

43. A cardboard box manufacturer needs to make open boxes from pieces of cardboard 12 in. square by cutting equal squares from the four corners and turning up the sides. Find the length of the side of the square to be cut out in order to obtain a box of the largest possible volume. What is the largest possible volume?

Solution:



$$\begin{aligned} \text{Volume is } V &= x \cdot (12 - 2x)^2 \\ &= x \cdot [2(6 - x)]^2 \\ &= 4x(6 - x)^2 \end{aligned}$$

To maximize volume, differentiate: $V = 4x(6 - x)^2$

$$\begin{aligned} \frac{dV}{dx} &= 4(6 - x)^2 + 4x(2)(6 - x)(-1) \\ &= 4(6 - x)^2 + (-8x)(6 - x) \\ &= (6 - x) \cdot 4 \cdot ((6 - x) - 2x) \\ &= 4(6 - x)(6 - 3x) \end{aligned}$$

Set $\frac{dV}{dx} = 0$

$$\begin{aligned} 4(6 - x)(6 - 3x) &= 0 \\ 6 - x = 0 &\text{ or } 6 - 3x = 0 \\ x = 6 &\text{ or } x = 2 \end{aligned}$$

$$\begin{aligned} \text{Check these: } V|_{x=6} &= 6 \cdot (12 - 2 \cdot 6)^2 = 6 \cdot (0)^2 = 0 \\ V|_{x=2} &= 2 \cdot (12 - 2 \cdot 2)^2 = 2 \cdot (12 - 4)^2 = 2 \cdot 8^2 = 128 \end{aligned}$$

Maximum volume of 128in^3 is achieved when the squares have sides of length 2.

44. A train leaves a station traveling north at the rate of 60 mph. One hour later, a second train leaves the same station traveling east at the rate of 45 mph. Find the rate at which the trains are separating 2 hours after the second train leaves the station.

Solution:

Let t be time in hours, after second train leaves station.
At $t = -1$, 1 hour before second train leaves station, first train leaves station going north
At $t = 0$, First train is at 60 miles N and second train leaves station going east
At $t > 0$, First train is $60 + 60t$ north, second train is $45t$ east
Let $x(t)$ and $y(t)$ be the distance of the eastbound and northbound trains from the station at time t. Then, respectively,

$$x(t) = 45t \quad \text{and} \quad y(t) = 60 + 60t.$$

The separation distance $s(t)$ at time t is

$$s(t) = \sqrt{(x(t))^2 + (y(t))^2}$$

So

$$\frac{ds}{dt} = \frac{1}{2\sqrt{(x(t))^2 + (y(t))^2}} \left[2x(t)\frac{dx}{dt} + 2y(t)\frac{dy}{dt}\right]$$

Because $x(2) = 90$, $y(2) = 180$, $\frac{dx}{dt} = 45$, and $\frac{dy}{dt} = 60$, we have

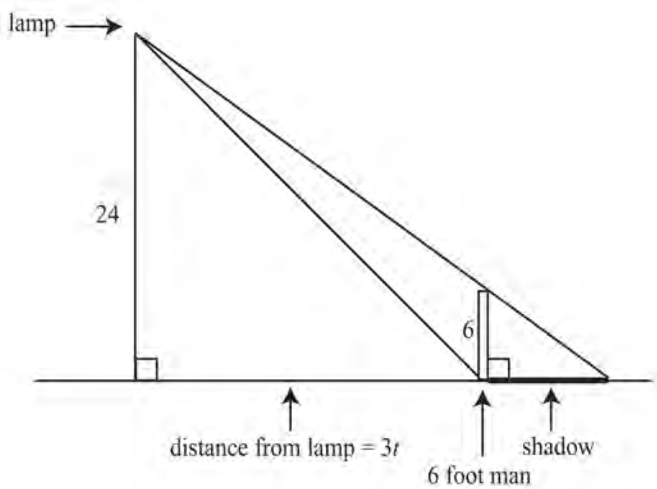
$$\begin{aligned}\frac{ds}{dt}(t = 2) &= \frac{1}{2\sqrt{(90)^2 + (180)^2}} [2(90)(45) + 2(180)(60)] \\ &= \frac{1}{180\sqrt{5}} [180(45 + 120)] \\ &= \frac{165}{\sqrt{5}} = 33\sqrt{5}\end{aligned}$$

Answer: $33\sqrt{5}$ mph is the rate of separation at the time two hours after the second train leaves the station.

45. A street light hangs on a pole 24 ft. above the sidewalk. A man 6 ft. tall walks away from the light at the rate of 3 ft/sec.

(a) At what rate is the length of his shadow increasing?

(b) At what rate is the tip of the shadow moving away from the pole?



Solution:

(a)
- Let s be the length of the man's shadow (in feet).
- Note that the two triangles are similar.
- Let t denote times in seconds. Then the distance from the point below the street lamp is given by $3t$.
- Since triangles are similar, $\dfrac{3t+s}{24} = \dfrac{s}{6}$

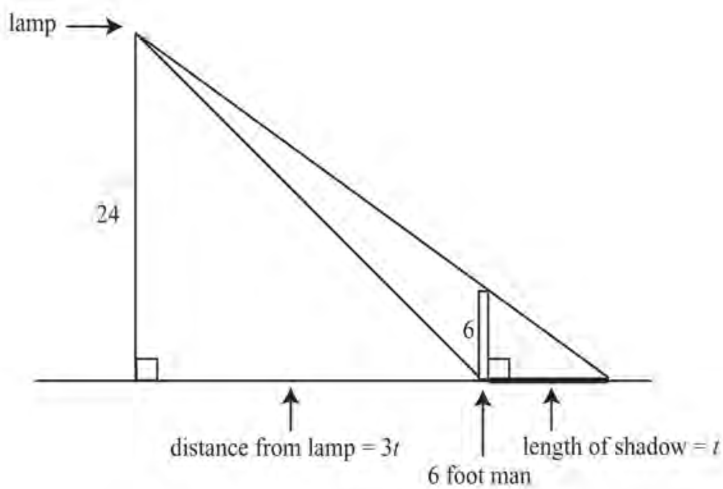
Solve for s:

$$\frac{3t}{24} + \frac{s}{24} = \frac{s}{6}$$
$$\frac{t}{8} = \frac{s}{6} - \frac{s}{24} = \frac{4s}{24} - \frac{s}{24} = \frac{3s}{24} = \frac{s}{8}$$
$$\frac{t}{8} = \frac{s}{8},\ t = s$$

The shadow is increasing at the rate $\dfrac{ds}{dt} = 1$, i.e. 1 foot/second.

(b) From above we have:

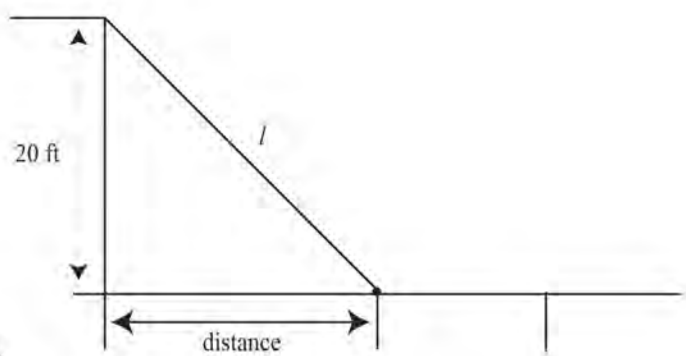
- The distance from the street lamp is given by $3t$, where t denotes the time in seconds.
- The length of the shadow is given by t.
- The distance from the pole to the tip of the shadow is $3t + t = 4t$.



Let the distance from the pole to the tip of the shadow be given by $x = 4t$. The distance is increasing at the rate $\frac{dx}{dt} = 4$, i.e. 4 feet/second.

46. A barge is pulled toward a dock by means of a taut cable. If the barge is 20 ft. below the level of the dock, and the cable is pulled in at the rate of 36 ft/min, find the speed of the barge when the cable is 52 ft. long.

Solution:



Let l be the length of the cable in feet.

We know $\frac{dl}{dt} = 36$ (in ft/min)

Also, it is known that the distance of the barge from the dock is given by (using the quadratic formula):

$$s = \sqrt{l^2 - (20)^2} = \sqrt{l^2 - 400} = (l^2 - 400)^{1/2}$$

The speed of the barge is given by

$$\frac{ds}{dt} = \frac{1}{2}(l^2 - 400)^{-1/2}(2l)\frac{dl}{dt}$$

When the cable is 52 ft. long ($l = 52$),

$$\begin{aligned}\frac{ds}{dt} =&\frac{1}{2}(52^2 - 400)^{-1/2}(2 \cdot 52) \cdot 36\\ =&\frac{1}{2}(2704 - 400)^{-1/2}(104) \cdot 36\\ =&\frac{1}{2}(2304)^{-1/2} \cdot 104 \cdot 36 = \frac{1872}{\sqrt{2304}} = \frac{1872}{48}\\ =&39\end{aligned}$$

So, the speed of the barge when the length of the rope is 52 ft. is 39 ft/min.

47. Let $y = 2x^2 - x$, $x = 2$, and $dx = \Delta x = 0.01$. Find the values of

   (a) Δy

   (b) dy

   Solution:

   (a)

$$\begin{aligned}\frac{dy}{dx} =&4x - 1\\ dy =&(4x - 1)\,dx\\ dy =&(4 \cdot 2 - 1) \cdot 0.01 = (8 - 1) \cdot 0.01 = 0.07\end{aligned}$$

   (b)

$$\begin{aligned}\Delta y =&y(2 + 0.01) - y(2)\\ =&y(2.01) - y(2)\\ =&\left[2(2.01)^2 - (2.01)\right] - \left[2(2)^2 - 2\right]\\ =&[2 \cdot 4.0401 - 2.01] - [8 - 2]\\ =&[8.0802 - 2.01] - 6\\ =&6.0702 - 6 = 0.0702\end{aligned}$$

48. Use differentials to approximate the maximum possible error that can be produced when calculating the volume of a cube if the length of an edge is known to be 2 ± 0.005 ft.

   Solution:

$$x =2,\ dx = \Delta x = \pm 0.005$$

$$\begin{aligned} y &= x^3 \\ y' &= 3x^2 \end{aligned}$$

$$\begin{aligned} \text{Maximum error } |\Delta y| \approx |dy| &= |y' \cdot dx| \\ &= |3x^2 \cdot dx| \\ &= |3(2)^2 \cdot (0.005)| = |12 \cdot 0.005| = 0.060 \end{aligned}$$

Maximum possible error is approximately ± 0.06 ft.

49. Use differentials to approximate $\sqrt{50}$.

Solution:

$$\begin{aligned} \text{Let } f(x) &= x^{1/2} \\ \text{Then } \frac{df}{dx} &= \frac{1}{2}x^{-1/2} \\ df &= \frac{1}{2}x^{-1/2}\,dx \end{aligned}$$

To approximate $\sqrt{50}$, let $x = 49$ and $dx = 1$.

$$\begin{aligned} \text{Then } f(x) &= f(49) = \sqrt{49} = 7 \\ \text{and } df &= \frac{1}{2}49^{-1/2} \cdot 1 \\ &= \frac{1}{2} \cdot \frac{1}{7} \cdot 1 = \frac{1}{14} \end{aligned}$$

$$\begin{aligned} \sqrt{50} &\approx f(49) + df \\ &= 7 + \frac{1}{14} \\ &\approx 7.07 \end{aligned}$$

50. The moment of inertia of an annular cylinder is $I = .5M(R_2^2 - R_1^2)$, where M is the mass of the cylinder, R_2 is its outer radius, and R_1 is its inner radius. Suppose that $R_1 = 2$, and R_2 changes from 4 to 4.01. Use differentials to estimate the resulting change in the moment of inertia.

Solution:

$$\frac{dI}{dR_2} = .5M(2R_2)$$

$$
\begin{aligned}
dI =& .5M(2R_2)\,dR_2 \\
\text{and } dR_2 =& 0.01,\ \ R_2 = 4 \\
\text{So: } dI =& .5M(2\cdot 4)(.01) \\
=& .5M(8)(.01) \\
=& 0.04M
\end{aligned}
$$

51. The range of a shell shot from a certain ship is $R = 300\sin(2\theta)$ meters, where θ is the angle above horizontal of the gun when it is shot. The gun is intended to be fired at an angle of $\pi/6$ radians to hit its target, but due to waves it actually shot .05 radians too low. Use differentials to estimate how far short of its target the shell will fall.

    Solution:

$$
\begin{aligned}
\theta =& \frac{\pi}{6} \\
d\theta =& -.05 \\
\frac{dR}{d\theta} =& 300\cos(2\theta)\cdot 2 = 600\cos(2\theta) \\
dR =& 600\cos(2\theta)d\theta \\
dR =& 600\cos\left(2\cdot\frac{\pi}{6}\right)\cdot(-.05) \\
=& 600\cos\frac{\pi}{3}\cdot(-.05) \\
=& 600\cos 60^\circ\cdot(-.05) \\
=& 600\cdot\frac{1}{2}\cdot(-.05) \\
=& -300\cdot .05 = -15
\end{aligned}
$$

    It will fall about 15 m short of the target.

52. In each of the following, determine whether the Intermediate Value Theorem guarantees the equation has a solution in the specified interval.

    (a) $x^5 + 2x^2 - 10x + 5 = 0,\ [1,2]$　　(b) $x - \sqrt{x} = 5,\ [4,9]$

    (c) $x = \cos x,\ [0,\pi]$　　(d) $x = \tan x,\ \left[\pi/4, 3\pi/4\right]$

    Solution:
    Intermediate Value Theorem:
    Suppose that f is continuous on the closed interval $[a,b]$ and let N be any number strictly between $f(a)$ and $f(b)$. Then there exists a number c in (a,b) such that $f(c) = N$.

(a)

$$\begin{aligned} x^5 + 2x^2 - 10x + 5 =& 0, \ [1, 2] \\ \text{Let } f(x) =& x^5 + 2x^2 - 10x + 5 \text{ Note } f \text{ is continuous} \\ f(1) =& 1^5 + 2 \cdot 1^2 - 10 \cdot 1 + 5 \\ =& 1 + 2 - 10 + 5 = -2 \\ f(2) =& 2^5 + 2 \cdot 2^2 - 10 \cdot 2 + 5 \\ =& 32 + 8 - 10 + 5 = 25 \end{aligned}$$

Now, $f(1) < 0 < f(2)$, so there is a c such that $f(c) = 0$, i.e. $x^5 + 2x^2 - 10x + 5 = 0$ is guaranteed a solution by Intermediate Value Theorem.

(b)

$$\begin{aligned} x - \sqrt{x} =& 5, \ [4, 9] \\ \text{Let } f(x) =& x - \sqrt{x} \\ f(4) =& 4 - \sqrt{4} = 4 - 2 = 2 \\ f(9) =& 9 - \sqrt{9} = 9 - 3 = 6 \end{aligned}$$

Since $f(x) = x - \sqrt{x}$ is continuous and $2 < 5 < 6$, Intermediate Value Theorem does guarantee a solution.

(c) $x = \cos x$, $[0, \pi]$
Let $f(x) = x - \cos x$. Note that $f(x)$ is continuous. Need c such that $f(c) = 0$.

$$\begin{aligned} \text{Now, } f(0) =& 0 - \cos 0 = -1 \\ f(\pi) =& \pi - \cos \pi = \pi - (-1) = \pi + 1 \end{aligned}$$

Now, $f(0) < 0 < f(\pi)$, so $f(x) = 0$ has a solution in $0, \pi$, so $x = \cos x$ has a solution in $[0, \pi]$, by Intermediate Value Theorem.

(d) $x = \tan x, \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$
Let $f(x) = x - \tan x$.
Recall that $\tan x$ has discontinuities at

$$\begin{aligned} &\frac{\pi}{2}, \ \frac{3\pi}{2}, \ \frac{5\pi}{2}, \text{ etc. and } -\frac{\pi}{2}, \ -\frac{3\pi}{2}, \text{ etc.} \\ \text{and } &\frac{\pi}{4} < \frac{2\pi}{4} < \frac{3\pi}{4} \text{ and } \frac{2\pi}{4} = \frac{\pi}{2}, \end{aligned}$$

so $f(x)$ has a discontinuity in the range indicated, so Intermediate Value Theorem does not apply.
Remark: This does not necessarily mean no solution exists, just that we aren't guaranteed one by Intermediate Value Theorem.

53. Suppose that $f''(x) = 4x + 3$, $f(1) = 2$, and $f'(1) = -3$. Find $f(x)$.

Solution:

$$\begin{aligned} f''(x) &= 4x + 3 \\ f'(x) &= \int f''(x)\,dx + C \\ &= \int (4x+3)\,dx + C \\ &= \frac{1}{2}4x^2 + 3x + C = 2x^2 + 3x + C \end{aligned}$$

$$\begin{aligned} \text{Find C: } f'(1) = \frac{1}{2}\cdot 4\cdot 1^2 + 3\cdot 1 + C &= -3 \\ \frac{1}{2}\cdot 4 + 3 + C &= -3 \\ 2 + 3 + C &= -3 \\ C &= -8 \end{aligned}$$

$$\begin{aligned} f'(x) &= 2x^2 + 3x - 8 \\ f(x) &= \int f'(x)\,dx + D \\ &= \int (2x^2 + 3x - 8)\,dx + D \\ &= \frac{2}{3}x^3 + \frac{3}{2}x^2 - 8x + D \\ f(1) &= 2 \\ \frac{2}{3}\cdot 1^3 + \frac{3}{2}\cdot 1^2 - 8\cdot 1 + D &= 2 \\ \frac{2}{3} + \frac{3}{2} - 8 + D &= 2 \\ D &= 2 + 8 - \frac{2}{3} - \frac{3}{2} \end{aligned}$$

$$\begin{aligned} D =& 10 - \frac{4}{6} - \frac{9}{6} = 10 - \frac{13}{6} \\ =& \frac{60}{6} - \frac{13}{6} = \frac{47}{6} \\ f(x) =& \frac{2}{3}x^3 + \frac{3}{2}x^2 - 8x + \frac{47}{6} \end{aligned}$$

54. Suppose that $f'(x) = x^2 + 3x + 2$, and $f(-3) = -\frac{3}{2}$. Find $f(x)$.
 Solution:

$$\begin{aligned} f(x) =& \int f'(x)\, dx + C \\ =& \int (x^2 + 3x + 2)\, dx + C \\ =& \frac{x^3}{3} + \frac{3x^2}{2} + 2x + C \end{aligned}$$

$$\begin{aligned} f(-3) = \frac{(-3)^3}{3} + \frac{3(-3)^2}{2} + 2(-3) + C =& -\frac{3}{2} \\ \frac{-27}{3} + \frac{27}{2} + (-6) + C =& -\frac{3}{2} \\ -9 + \frac{27}{2} - 6 + C =& -\frac{3}{2} \\ -15 + \frac{27}{2} + C =& -\frac{3}{2} \\ -\frac{30}{2} + \frac{27}{2} + C =& -\frac{3}{2} \\ -\frac{3}{2} + C =& -\frac{3}{2} \\ C =& 0 \\ f(x) =& \frac{x^3}{3} + \frac{3x^2}{2} + 2x \end{aligned}$$

55. Let $f(t)$ be the function defined by the graph shown.

 Estimate the following to the nearest integer.

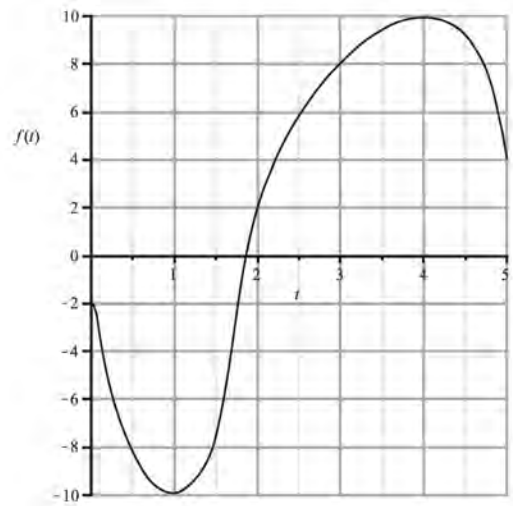
 (a) The instantaneous rate of change of f at $t = 3$.

 (b) The average rate of change of f over the interval [0, 4].

 (c) The intervals where $f(t)$ is increasing and where it is decreasing.

(d) The inflection point or points of $f(t)$.

(e) The intervals where $f'(t)$ is increasing and where it is decreasing.



Solution:

(a) 4

(b) 3

(c) $f(t)$ is increasing in $[1, 4]$ and decreasing in $[0, 1]$ and $[4, 5]$

(d) $(2, 2)$

(e) $f'(t)$, the derivative of the function shown, is increasing on $[0, 2]$ and is decreasing in $[2, 5]$, as can be seen by noting the concavity of the function, which indicates the sign of the derivative of $f'(t)$.

56. Let $v(t)$, as shown in the graph, be the velocity of a car in meters per second at time t in seconds, where positive velocity means the car is moving forward.

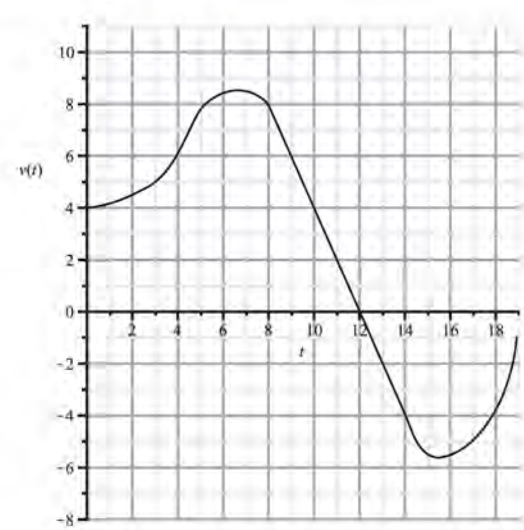
Estimate to the nearest integer.

(a) When the car stopped?

(b) How far the car traveled in the time interval 8 to 12 seconds?

(c) How far the car traveled in the time interval 12 to 14 seconds?

(d) How far the car traveled in the time interval 8 to 14 seconds?



Solution:

(a) When time is 12 seconds

(b) 16 meters

(c) 4 meters

(d) 20 meters (Note: The total *displacement* is approximately 12 meters.)

57. Consider the car whose motion is described in Problem 56.

(a) Is the car moving forward or backward at the time 2 seconds?

(b) Is the driver's foot on the gas or the brake at 2 seconds?

(c) Is the car moving forward or backward at the time 16 seconds?

(d) Is the driver's foot on the gas or the brake at 16 seconds?

Solution

(a) Forward

(b) Foot on the gas

(c) Backward

(d) Foot on the brake

58. Integrate the following:

(a) $\int_0^1 x(3x^2+4)^4\,dx$

(b) $\int (2x^4+4x)^3(2x^3+1)\,dx$

(c) $\int_1^2 \frac{(2x^3+3x+5)}{x^5}\,dx$

(d) $\int \frac{(3x^2+6x+2)}{x^4}\,dx$

(e) $\int_3^8 \sqrt{12-x}\,dx$

(f) $\int (3x-1)\sqrt{3x^2-2x+1}\,dx$

(g) $\int \frac{1}{(x+5)^4}\,dx$

(h) $\int \frac{(5x+\frac{1}{2})}{(10x^2+2x+40)^4}\,dx$

(i) $\int 3x^2\sqrt{6x^3+1}\,dx$

(j) $\int x\sqrt{2-x}\,dx$

(k) $\int \frac{\sqrt[3]{3+z^{-1}}}{z^2}\,dz$

(l) $\int \sec x \tan x \cos(\sec x)\,dx$

(m) $\int \sin\frac{x}{3}\left(\cos\frac{x}{3}\right)^3\ dx$

(n) $\int \tan x\ (\sec x)^2\ dx$

Solution:

(a)

$$\begin{aligned}\int_0^1 x(3x^2+4)^4\ dx &= \int_4^7 \frac{1}{6}\,du\left[u^4\right]\\ &= \frac{1}{6}\int_4^7 u^4\ du\\ &= \frac{1}{6}\left[\frac{u^5}{5}\right]_4^7\\ &= \frac{1}{6}\left[\frac{7^5}{5}-\frac{4^5}{5}\right]\\ &= \frac{1}{6}\left[\frac{16807-1024}{5}\right]\\ &= \frac{15783}{30}=\frac{5261}{10}\end{aligned}$$

$$\begin{aligned}\text{Let } u &= 3x^2+4\\ du &= 6x\ dx\\ \frac{1}{6}\,du &= x\ dx\end{aligned}$$

(b)

$$\begin{aligned}\int (2x^4+4x)^3(2x^3+1)\,dx &= \int \frac{1}{4}(u^3)\,du \\ &= \frac{1}{4}\int u^3\,du \\ &= \frac{1}{4}\left[\frac{u^4}{4}+C\right] \\ &= \frac{1}{4}\left[\frac{(2x^4+4x)^4}{4}+C\right] \\ &= \frac{(2x^4+4x)^4}{16}+C\end{aligned}$$

$$\begin{aligned}\text{Let } u &= 2x^4+4x \\ du &= (8x^3+4)\,dx \\ du &= 4(2x^3+1)\,dx \\ \frac{1}{4}\,du &= (2x^3+1)\,dx\end{aligned}$$

(c)

$$\begin{aligned}\int_1^2 \frac{(2x^3+3x+5)}{x^5}\,dx &= \int_1^2 \left[\frac{2x^3}{x^5}+\frac{3x}{x^5}+\frac{5}{x^5}\right]dx \\ &= 2\int \frac{1}{x^2}\,dx + 3\int \frac{1}{x^4}\,dx + 5\int \frac{1}{x^5}\,dx \\ &= 2\int x^{-2}\,dx + 3\int x^{-4}\,dx + 5\int x^{-5}\,dx \\ &= 2\left[\frac{x^{-1}}{-1}+c_1\right]+3\left[\frac{x^{-3}}{-3}+c_2\right]+5\left[\frac{x^{-4}}{-4}+c_3\right] \\ \text{Let } c_1+c_2+c_3 &= C \\ &= \frac{-2}{x}-\frac{1}{x^3}-\frac{5}{4x^4}+C \\ &= \left[\frac{-2}{x}-\frac{1}{x^3}-\frac{5}{4x^4}\right]_1^2 \\ &= \left[\frac{-2}{2}-\frac{1}{(2)^3}-\frac{5}{4(2)^4}\right]-\left[\frac{-2}{1}-\frac{1}{(1)^3}-\frac{5}{4(1)^4}\right] \\ &= -1-\frac{1}{8}-\frac{5}{64}+2+1+\frac{5}{4} \\ &= \frac{195}{64}\end{aligned}$$

(d)

$$\int \frac{(3x^2+6x+2)}{x^4}\,dx = \int \left[\frac{3x^2}{x^4}+\frac{6x}{x^4}+\frac{2}{x^4}\right]dx$$

$$
\begin{aligned}
&= 3\int \frac{1}{x^2}\,dx + 6\int \frac{1}{x^3}\,dx + 2\int \frac{1}{x^4}\,dx \\
&= 3\int x^{-2}\,dx + 6\int x^{-3}\,dx + 2\int x^{-4}\,dx \\
&= 3\left[\frac{x^{-1}}{-1} + c_1\right] + 6\left[\frac{x^{-2}}{-2} + c_2\right] + 2\left[\frac{x^{-3}}{-3} + c_3\right]
\end{aligned}
$$

Let $c_1 + c_2 + c_3 = C$

$$
= \frac{-3}{x} - \frac{3}{x^2} - \frac{2}{3x^3} + C
$$

(e)

$$
\begin{aligned}
\int_3^8 \sqrt{12 - x}\,dx &= \int_9^4 (-du)(u)^{1/2} \\
&= -\int_9^4 u^{1/2}\,du \\
&= -\left[\frac{2u^{3/2}}{3}\right]_4^9 \\
&= -\left[\frac{2(4)^{3/2}}{3} - \frac{2(9)^{3/2}}{3}\right] \\
&= -\frac{2(2)^3}{3} + \frac{2(3)^3}{3} \\
&= -\frac{2(8)}{3} + \frac{2(27)}{3} \\
&= \frac{-16 + 54}{3} = \frac{38}{3}
\end{aligned}
$$

Let $u = 12 - x$

$du = -dx$

$-du = dx$

(f)

$$
\begin{aligned}
\int (3x - 1)\sqrt{3x^2 - 2x + 1}\,dx &= \int \frac{1}{2} du (u)^{1/2} \\
&= \frac{1}{2}\int u^{1/2}\,du \\
&= \frac{1}{2}\left[\frac{2u^{3/2}}{3} + C\right] \\
&= \frac{1}{2}\left[\frac{2(3x^2 - 2x + 1)^{3/2}}{3} + C\right] \\
&= \frac{(3x^2 - 2x + 1)^{3/2}}{3} + C \\
&= \frac{1}{3}(3x^2 - 2x + 1)^{3/2} + C
\end{aligned}
$$

Let $u = 3x^2 - 2x + 1$

$du = (6x - 2)\,dx$

$du = 2(3x - 1)\,dx$

$\frac{1}{2} du = (3x - 1)\,dx$

(g)

$$\int \frac{1}{(x+5)^4}\,dx = \int \frac{1}{u^4} \qquad \text{Let } u = x+5$$
$$du = dx$$

$$\begin{aligned}
&= \int u^{-4} \\
&= \frac{u^{-3}}{-3} + C \\
&= \frac{(x+5)^{-3}}{-3} + C \\
&= -\frac{1}{3}\left(\frac{1}{x+5}\right)^3 + C \\
&= \frac{-1}{3(x+5)^3} + C
\end{aligned}$$

(h)

$$\begin{aligned}
\text{Let } u &= 10x^2 + 2x + 10 \\
du &= (20x+2)\,dx \\
du &= 2(10x+2)\,dx \\
du &= 4(5x+\frac{1}{2})\,dx \\
\frac{1}{4}du &= (5x+\frac{1}{2})\,dx
\end{aligned}$$

$$\begin{aligned}
\int \frac{\left(5x+\frac{1}{2}\right)}{(10x^2+2x+40)^4}\,dx &= \int \frac{1}{4}du(u)^{-4} \\
&= \frac{1}{4}\int u^{-4}\,du \\
&= \frac{1}{4}\left[\frac{u^{-3}}{-3} + C\right] \\
&= \left[\frac{-1}{12(10x^2+2x+40)^3} + C\right]
\end{aligned}$$

(i)

$$\begin{aligned}
\text{Let } u &= 6x^3 + 1 \\
du &= 18x^2\,dx \\
du &= 6(3x^2)\,dx \\
\frac{1}{6}\,du &= 3x^2\,dx
\end{aligned}$$

$$\begin{aligned}
\int 3x^2\sqrt{6x^3+1}\,dx &= \int \frac{1}{6}\,du(u)^{1/2} \\
&= \frac{1}{6}\int u^{1/2}\,du \\
&= \frac{1}{6}\left[\frac{2u^{3/2}}{3} + C\right] \\
&= \frac{1}{9}\left(6x^3+1\right)^{3/2} + C
\end{aligned}$$

(j)

$$\begin{aligned}\int x\sqrt{2-x}\,dx &= \int (2-u)(u)^{1/2}(-du)\\ &= \int (u-2)(u)^{1/2}\,du\\ &= \int (u^{3/2}-2u^{1/2})du\\ &= \frac{2u^{5/2}}{5} - 2\left(\frac{2}{3}u^{3/2}\right) + C\\ &= \frac{2(2-x)^{5/2}}{5} - \frac{4}{3}(2-x)^{3/2} + C\end{aligned}$$

$$\begin{aligned}\text{Let } u &= 2-x\\ x &= 2-u\\ du &= -\,dx\\ -du &= dx\end{aligned}$$

(k)

$$\begin{aligned}\int \frac{\sqrt[3]{3+z^{-1}}}{z^2}\,dz &= \int u^{1/3}(-du)\\ &= \int -u^{1/3}\,du\\ &= -\int u^{1/3}\,du\\ &= -\left[\frac{3u^{4/3}}{4} + C\right]\\ &= -\frac{3}{4}(3+z^{-1})^{4/3} + C\end{aligned}$$

$$\begin{aligned}\text{Let } u &= 3+z^{-1}\\ du &= -\,z^{-2}\,dz\\ -du &= \frac{dz}{z^2}\end{aligned}$$

(l)

$$\begin{aligned}\int \sec x\tan x\cos(\sec x)\,dx &= \int \cos(u)\,du\\ &= \sin(u) + C\\ &= \sin(\sec x) + C\end{aligned}$$

$$\begin{aligned}\text{Let } u &= \sec x\\ du &= \sec x\tan x\end{aligned}$$

(m)

$$\int \sin\frac{x}{3}\left(\cos\frac{x}{3}\right)^3\,dx = \int u^3(-3du) \qquad \text{Let } u = \cos\left(\frac{x}{3}\right)$$
$$= -3\int u^3\,du \qquad du = -\sin\left(\frac{x}{3}\right)\left[\frac{1}{3}\right]dx$$
$$= -3\left[\frac{u^4}{4} + C\right] \qquad -3\,du = \sin\left(\frac{x}{3}\right)\,dx$$
$$= \frac{-3\left[cos\left(\frac{x}{3}\right)\right]^4}{4} + C$$
$$= \frac{-3}{4}\left[cos\left(\frac{x}{3}\right)\right]^4 + C$$

(n)

$$\int \tan x\,(\sec x)^2\,dx = \int u\,du \qquad \text{Let } u = \tan x$$
$$= \left[\frac{u^2}{2} + C\right] \qquad du = (\sec x)^2\,dx$$
$$= \frac{(\tan x)^2}{2} + C$$

Alternatively,

$$\int \tan x\,(\sec x)^2\,dx = \int u\,du \qquad \text{Let } u = \sec x$$
$$= \left[\frac{u^2}{2} + C\right] \qquad du = \sec x\tan x\,dx$$
$$= \frac{(\sec x)^2}{2} + C$$

59. Differentiate the following:

(a) $f(x) = \int_3^x (t^2 - 8)\,dt$

(b) $f(x) = \int_4^{x^2} (t^2 + 6)^{5/2}\,dt$

Solution:

(a) $Dx \int_3^x (t^2 - 8)dt = x^2 - 8$

(b) $Dx \int_4^{x^2} (t^2 + 6)^{5/2}dt = \left((x^2)^2 + 6\right)^{5/2} 2x = 2x(x^4 + 6)^{5/2}$

60. Find the area of the region bounded by the graphs of $y = x^2$ and $y = x^4$.

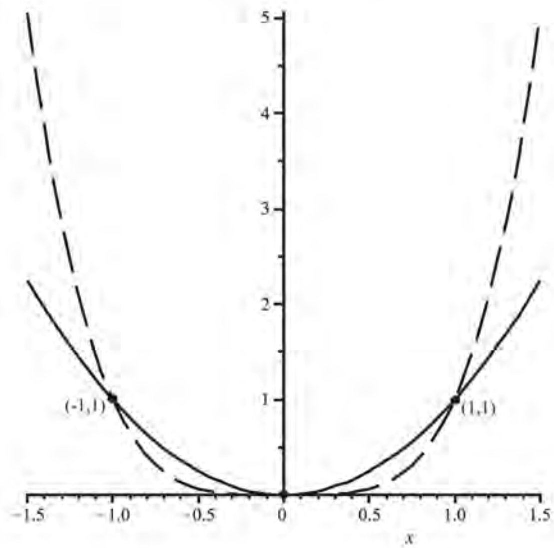
Solution:

$$y = x^2$$
$$y = x^4$$

Points of Intersection:

$$x^4 = x^2$$
$$x^2(x^2 - 1) = 0$$
$$x^2 = 0, \ (x-1)(x+1) = 0$$
$$x = 0, \ x = 1, -1$$

They are: $(0, 0)$, $(1, 1)$, $(-1, 1)$



$$\text{Area} = 2\int_0^1 (x^2 - x^4)\, dx = 2\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1$$
$$= 2\left[\frac{1}{3} - \frac{1}{5}\right]$$
$$= 2\left[\frac{5-3}{15}\right]$$
$$= 2\left[\frac{2}{15}\right]$$
$$= \frac{4}{15}$$

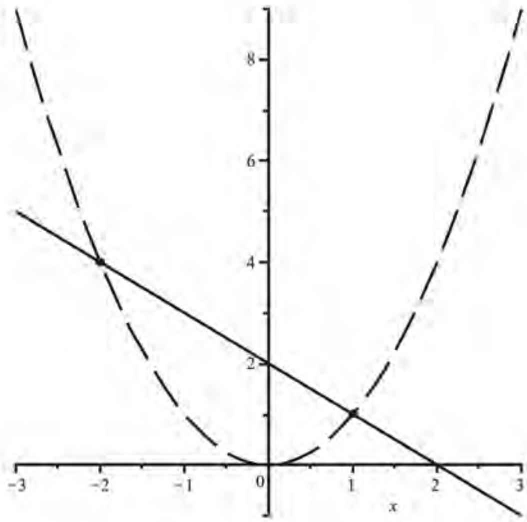
61. Find the area of the region bounded by the graphs of $y = x^2$ and $y = 2 - x$.

Solution:

$$y = x^2$$
$$y = 2 - x$$

Points of Intersection:

$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2, x = 1$$



$$\begin{aligned}
A &= \int_{-2}^{1} (2 - x) - x^2 dx \\
&= \left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-2}^{1} \\
&= \left(2 - \frac{1}{2} - \frac{1}{3}\right) - \left(2(-2) - \frac{1}{2}(-2)^2 - \frac{1}{3}(-2)^3\right) \\
&= \left(\frac{12}{6} - \frac{3}{6} - \frac{2}{6}\right) - \left(-4 - 2 + \frac{8}{3}\right) \\
&= \frac{12}{6} - \frac{3}{6} - \frac{2}{6} + \frac{36}{6} - \frac{16}{6} \\
&= \frac{27}{6} = \frac{9}{2}
\end{aligned}$$

62. Find the area of the region bounded by the graphs of $y = x^4$ and $y = 8x$.

Solution:

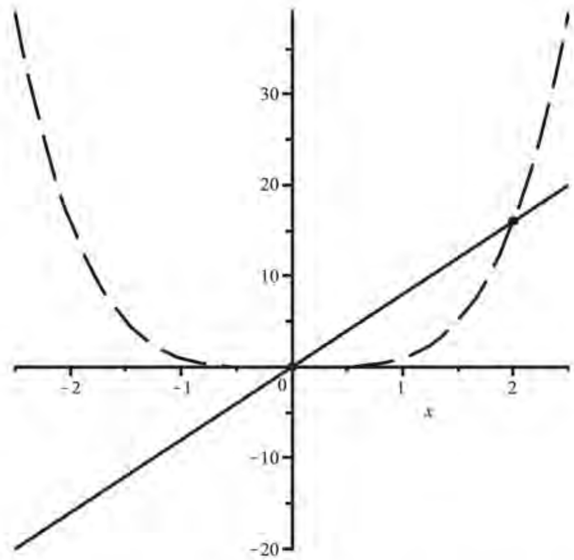
$$y = 8x$$
$$y = x^4$$

Points of Intersection:

$$x(x^3 - 8) = 0$$
$$x(x-2)(x^2 + 2x + 4) = 0$$
$$x = 0, x = 2$$

$$A = \int_0^2 (8x - x^4)dx$$
$$= \left[4x^2 - \frac{1}{5}x^5\right]_0^2$$
$$= 16 - \frac{32}{5} = \frac{48}{5}$$

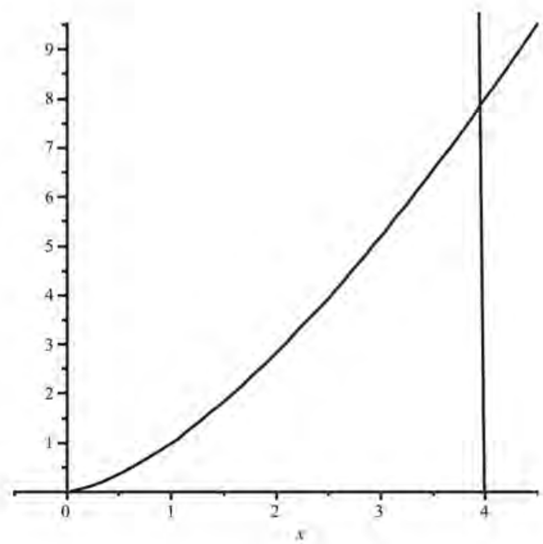
$$= \frac{48}{5}$$



63. Find the volume of the solid generated by revolving about the x-axis the region bounded by the graph of $y = x^{3/2}$, the x-axis, and the line $x = 4$.

Solution:

$$V = \pi \int_a^b \left[f(x)\right]^2 dx$$
$$V = \pi \int_0^4 x^3 dx$$
$$= \left[\frac{\pi}{4}x^4\right]_0^4$$
$$= 64\pi$$



64. Find the volume of the solid generated by revolving about the x-axis the region bounded by

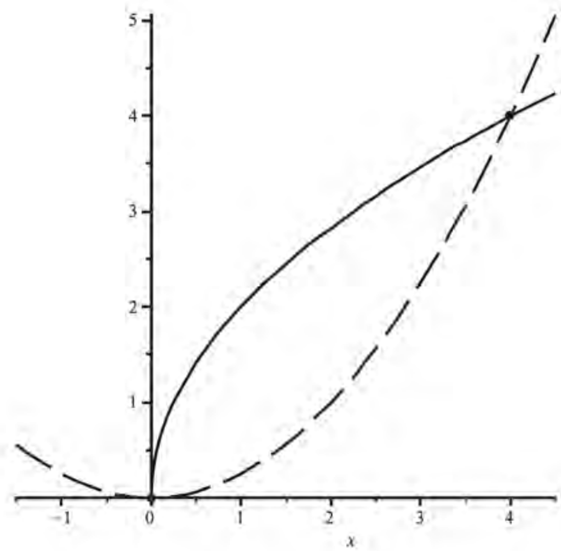
$y^2 = 4x$ and $x^2 = 4y$.

Solution:

$$y = \frac{1}{4}x^2$$
$$x = \frac{1}{4}y^2$$

Points of Intersection:

$$x = \frac{1}{64}x^4$$
$$x\left(\frac{1}{64}x^3 - 1\right) = 0$$
$$x = 0, x = 4$$



$$V = \pi \int_a^b \left[f(x)\right]^2 - \left[g(x)\right]^2 dx$$
$$V = \pi \int_0^4 \left(\left(2\sqrt{x}\right)^2 - \left(\frac{1}{4}x^2\right)^2\right) dx$$
$$V = \pi \int_0^4 \left(4x - \frac{1}{16}x^4\right) dx$$
$$= \pi \left[2x^2 - \frac{x^5}{80}\right]_0^4$$
$$= \left(32 - \frac{64}{5}\right)\pi$$
$$= \frac{96\pi}{5}$$

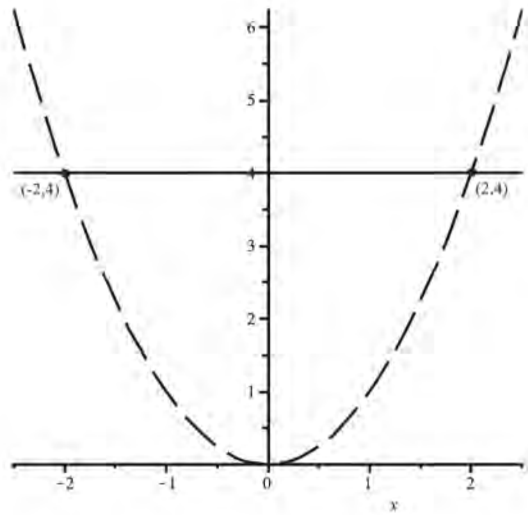
65. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the graph of $y = x^2$, $y = 4$, and the y-axis about a) the y-axis; b) $y = 4$.

Solution:

$$y = 4$$
$$y = x^2$$

Points of Intersection:

$$x^2 - 4 = 0$$
$$x = \pm 2$$



(a) About the y-axis:

$$\begin{aligned} V &= 2\pi \int_0^2 x\left[4 - x^2\right] dx \\ &= 2\pi \int_0^2 \left(4x - x^3\right) dx \\ &= 2\pi \left[2x^2 - \frac{1}{4}x^4\right]_0^2 \\ &= 2\pi(8 - 4) \\ &= 8\pi \end{aligned}$$

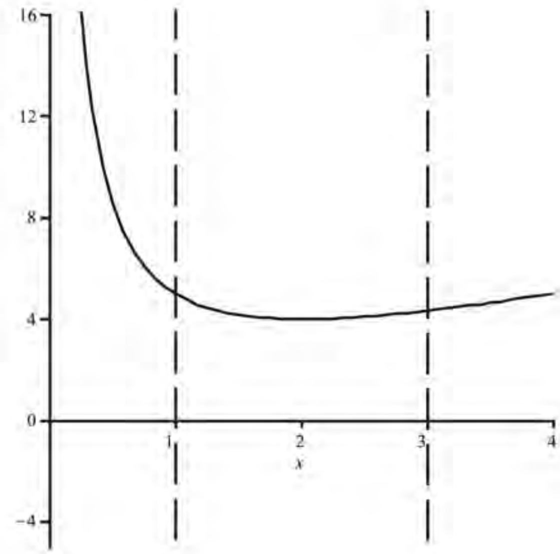
(b) About the axis where $y = 4$:

$$\begin{aligned} 2\pi \int_0^4 (4 - y)\sqrt{y}\, dy &= 2\pi \int_0^4 (4y^{1/2} - y^{3/2}) dy \\ &= 2\pi \left[\frac{4(2)y^{3/2}}{3} - \frac{2y^{5/2}}{5}\right]_0^4 \\ &= 2\pi \left[\frac{8(4)^{3/2}}{3} - \frac{2(4)^{5/2}}{5} - 0\right] \\ &= 2\pi \left[\frac{8(8)}{3} - \frac{2(2)^5}{5}\right] \\ &= 2\pi \left[\frac{2^6}{3} - \frac{2^6}{5}\right] \end{aligned}$$

$$=\frac{2^7\pi}{15}[5-3]$$
$$=\frac{2^8\pi}{15}=\frac{256\pi}{15}$$

66. Find the volume of the solid generated by revolving about the y-axis the region bounded by $y = x + \frac{4}{x}$, the x-axis, and the lines $x = 1$ and $x = 3$.

Solution:



$$V = 2\pi \int_a^b x\left[f(x)\right]dx$$
$$V = 2\pi \int_1^3 x\left(x+\frac{4}{x}\right)dx$$
$$= 2\pi \int_1^3 \left(x^2+4\right)dx$$
$$= 2\pi \left[\left(\frac{x^3}{3}+4x\right)\right]_1^3$$
$$= 2\pi \left[\left(\frac{27}{3}+12\right)-\left(\frac{1}{3}+4\right)\right]$$
$$= 2\pi \left[\frac{26}{3}+8\right]$$
$$= 2\pi \left[\frac{26+24}{3}\right]$$
$$= 2\pi \left(\frac{50}{3}\right)=\frac{100\pi}{3}$$

67. A solid has for its base the region in the first quadrant bounded by $x^2 + y^2 = 25$. Every plane section of the solid taken perpendicular to the x-axis is a square. Find the volume of the solid.

Solution:

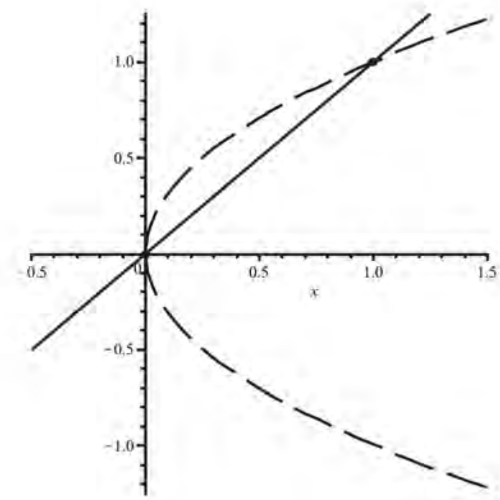
$$A = y \cdot y$$

$$x^2 + y^2 = 25$$
$$y = \sqrt{25 - x^2}$$

$$
\begin{aligned}
V &= \int_{x=0}^{x=5} y^2 dx \\
&= \int_0^5 (25 - x^2) dx \\
&= \left[25x - \frac{x^3}{3} \right]_0^5 \\
&= 125 - \frac{125}{3} \\
&= \frac{375 - 125}{3} \\
&= \frac{250}{3}
\end{aligned}
$$

68. A solid has as its base the region in the xy-plane bounded by the graphs of $y = x$ and $y^2 = x$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a semicircle with diameter in the xy-plane.

Solution:



$$
\begin{aligned}
y &= x \\
y^2 &= x
\end{aligned}
$$

Points of Intersection:

$$(0, 0), (1, 1)$$

For any x, diameter of the semicircle is:

$$
\begin{aligned}
d &= \sqrt{x} - x \\
\Rightarrow \text{radius} &= \frac{\sqrt{x} - x}{2}
\end{aligned}
$$

$$
\begin{aligned}
A &= \frac{1}{2}\pi \left(\frac{\sqrt{x} - x}{2} \right)^2 \\
V &= \int_a^b A(x) dx
\end{aligned}
$$

$$
\begin{aligned}
V &= \frac{1}{8}\pi \int_0^1 \left(\sqrt{x} - x\right)^2 dx \\
&= \frac{\pi}{8} \int_0^1 \left(x - 2x^{3/2} + x^2\right) dx \\
&= \left[\frac{\pi}{8}\left(\frac{x^2}{2} - \frac{4}{5}x^{5/2} + \frac{x^3}{3}\right)\right]_0^1 \\
&= \frac{\pi}{8}\left(\frac{1}{2} - \frac{4}{5} + \frac{1}{3}\right) = \frac{\pi}{240}
\end{aligned}
$$

69. Find the average value of the function $f(x) = x^2 + x + 1$ on the interval $[-1, 2]$.

Solution:

$$
\begin{aligned}
\frac{1}{2-(-1)} \int_{-1}^{2} (x^2 + x + 1)dx &= \frac{1}{3} \int_{-1}^{2} (x^2 + x + 1)dx \\
&= \frac{1}{3}\left[\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{2} \\
&= \frac{1}{3}\left[\left(\frac{8}{3} + \frac{4}{2} + 2\right) - \left(-\frac{1}{3} + \frac{1}{2} - 1\right)\right] \\
&= \frac{1}{3}\left[\frac{8}{3} + \frac{1}{3} + 2 + 2 - \frac{1}{2} + 1\right] \\
&= \frac{1}{3}\left[8 - \frac{1}{2}\right] = \frac{1}{3}\left[\frac{15}{2}\right] = \frac{15}{6} = \frac{5}{2}
\end{aligned}
$$

70. Find the average value of the function $f(x) = \sin x$ on the interval $[0, \pi]$.

Solution:

$$
\begin{aligned}
\frac{1}{\pi - 0} \int_0^{\pi} \sin^2 x \, dx &= \frac{1}{\pi} \int_0^{\pi} \frac{1 - \cos 2x}{2} \, dx \\
&= \frac{1}{2\pi}\left[x - \frac{1}{2}\sin(2x)\right]_0^{\pi} \\
&= \frac{1}{2\pi}\left[\left(\pi - \frac{1}{2}\sin(2\pi)\right) - \left(0 - \frac{1}{2}\sin(0)\right)\right] \\
&= \frac{1}{2\pi}\left[\pi - 0 - 0 + 0\right] \\
&= \frac{1}{2}
\end{aligned}
$$

71. A cylindrical water tank with a circular base has radius 3 feet and height 10 feet. How much work is required to empty the tank by pumping the water out of the top if a) the tank is full? b) the tank is half full? (Assume that the density of water is 62.5 lb/ft^3.)

Solution:

(a) $F = \text{(density)(volume)}$
$F = \text{(density)(area thickness)}$
$F = (62.5)\left(\pi(3)^2\right) y$
$F = 562.5\pi y$
$W = (F)\text{(distance)}$

$$\begin{aligned} W &= \int_0^{10} 562.5\pi(10-y)\,dy \\ &= 562.5\pi \int_0^{10} (10-y)\,dy \\ &= 562.5\pi \left[10y - \frac{y^2}{2}\right]_0^{10} \\ &= 562.5\pi \left[10(10) - \frac{10^2}{2} - \left(10(0) - \frac{0^2}{2}\right)\right] \\ &= 562.5\pi \left[100 - \frac{100}{2}\right] = 562.5\pi(50) = 28125\pi \end{aligned}$$

(b)

$$\begin{aligned} W &= \int_0^{5} 562.5\pi(10-y)\,dy \\ &= 562.5\pi \int_0^{5} (10-y)\,dy \\ &= 562.5\pi \left[10y - \frac{y^2}{2}\right]_0^{5} \\ &= 562.5\pi \left[10(5) - \frac{5^2}{2} - \left(10(0) - \frac{0^2}{2}\right)\right] \\ &= 562.5\pi \left[50 - \frac{25}{2}\right] \\ &= 562.5\pi \left[\frac{100-25}{2}\right] = 562.5\pi\left(\frac{75}{2}\right) = 562.5\pi(37.5) = 21093.75\pi \end{aligned}$$

72. A bucket with 24 lb of water is raised 30 feet from the bottom of a well. Find the work done assuming that a) the weight of the empty bucket is 4 lb and the weight of the rope is negligible; b) the bucket weighs 4 lb and the rope weighs 4 oz/ft; c) in addition, the water leaks out of the bucket at a constant rate and that only 18 lb are left at the top.

Solution:

(a)

$$\begin{aligned} F &= \text{(weight of water + weight of bucket)} \\ &= 24 + 4 \\ &= 28 \\ W &= \int_0^{30} 28\, dx \\ &= [28x]_0^{30} \\ &= 28(30) = 840 \end{aligned}$$

(b)

$$\begin{aligned} F &= \text{(weight of water + weight of bucket + weight of rope)} \\ &= 24 + 4 + \frac{1}{4}x \\ &= 28 + \frac{1}{4}x \\ W &= \int_0^{30} \left(28 + \frac{1}{4}x\right) dx \\ &= \left[28x + \frac{1}{8}x^2\right]_0^{30} \\ &= 28(30) + \frac{1}{8}(30)^2 = 840 + 112.5 = 952.5 \end{aligned}$$

(c)

$$\begin{aligned} F &= \text{(weight of water + weight of bucket + weight of rope)} - \text{(weight leaking out)} \\ &= 24 + 4 + \frac{1}{4}x - \frac{6}{30}x \\ &= 28 + \frac{1}{20}x \\ W &= \int_0^{30} \left(28 + \frac{1}{20}x\right) dx \end{aligned}$$

$$= \left[28x + \frac{1}{40}x^2 \right]_0^{30}$$
$$= 28(30) + \frac{1}{40}(30)^2 = 840 + 22.5 = 862.5$$

73. Match each numbered item with a lettered item. (There are more lettered items than numbered items. Some lettered items don't match any numbered item.)

1. Definition of $\lim_{x \to a} f(x) = L$.
2. Definition of $\lim_{x \to a^+} f(x) = L$.
3. Definition of $\lim_{x \to \infty} f(x) = L$.
4. Definition of "f is continuous at a".
5. The Intermediate Value Theorem.
6. Definition of the derivative of f at a.
7. Definition of a differentiable function f at a.
8. Theorem relating differentiability and continuity.
9. The power rule for differentiation.
10. Definition of the differential.
11. Definition of a function f having an absolute maximum at c.
12. Definition of a function f having a local maximum at c.
13. The Extreme Value Theorem.
14. Definition of a function that is increasing on an interval I.
15. The Mean Value Theorem.
16. Definition of an antiderivative of f on an interval I.

A. $\lim_{x \to a} f(x) = f(a)$, $\lim_{x \to a} f(x)$ and $f(a)$ both exist.

B. For every $\epsilon > 0$ there is a corresponding number N such that $|f(x) - L| < \epsilon$ whenever $x > N$.

C. If f is continuous on the closed interval $[a, b]$, and N is a number strictly between $f(a)$ and $f(b)$, then there exists a number c in (a, b) such that $f(c) = N$.

D. The limit $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists.

E. If f is differentiable at a, then f is continuous at a.

F. If f is continuous at a, then f is differentiable at a.

G. $\frac{d}{dx}x^n = nx^{n-1}$.

H. The function g has the property $g'(x) = f(x)$ for all x in I.

I. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

J. $f(c) \geq f(x)$ for all x in the domain of f.

K. For every $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $a < x < a + \delta$.

L. For every $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

M. There is an open interval I containing c such that $f(c) \geq f(x)$ for all x in I.

N. $f'(x) > 0$ for all x in I.

O. If f is continuous on $[a, b]$ then there are numbers c and d in $[a, b]$ such that $f(c)$ is an absolute maximum for f in $[a, b]$ and $f(d)$ is an absolute minimum for f in $[a, b]$.

P. If f is differentiable, $dy = f'(x)dx$.

Q. $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in I.

R. If f is continuous on $[a, b]$ and differentiable in (a, b), then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Solution:

1. L
2. K
3. B
4. A
5. C
6. I
7. D
8. E
9. G
10. P
11. J
12. M
13. O
14. Q
15. R
16. H