

Derivatives

D_x e^x = e^x
D_x sin(x) = cos(x)
D_x cos(x) = -sin(x)
D_x tan(x) = sec^2(x)
D_x cot(x) = -csc^2(x)
D_x sech(x) = sec(x) tanh(x)
D_x csch(x) = -csch(x) coth(x)
D_x sin^-1(x) = 1/sqrt(1-x^2), x in [-1,1]
D_x cos^-1(x) = -1/sqrt(1-x^2), x in [-1,1]
D_x tan^-1(x) = 1/(1+x^2), x in (-inf, inf)
D_x sec^-1(x) = 1/(|x|sqrt(x^2-1)), |x| > 1
D_x sinh(x) = cosh(x)
D_x cosh(x) = sinh(x)
D_x tanh(x) = sech^2(x)
D_x coth(x) = -csch^2(x)
D_x sech(x) = -sech(x) tanh(x)
D_x csch(x) = -csch(x) coth(x)
D_x sinh^-1(x) = 1/sqrt(x^2+1)
D_x cosh^-1(x) = 1/sqrt(x^2-1), x > 1
D_x tanh^-1(x) = 1/(1-x^2), -1 < x < 1
D_x sech^-1(x) = -1/(sqrt(1-x^2)), 0 < x < 1
D_x ln(x) = 1/x

Integrals

int 1/x dx = ln|x| + c
int e^x dx = e^x + c
int a^x dx = 1/a * a^x + c
int e^(ax) dx = 1/a * e^(ax) + c
int 1/(1+x^2) dx = tan^-1(x) + c
int 1/(1-x^2) dx = tan^-1(x) + c
int 1/(x^2+1) dx = sec^-1(x) + c
int sinh(x) dx = cosh(x) + c
int cosh(x) dx = sinh(x) + c
int tanh(x) dx = ln|cosh(x)| + c
int tanh(x) sech(x) dx = -sech(x) + c
int sech^2(x) dx = tanh(x) + c
int cosh(x) coth(x) dx = sech(x) + c
int tan(x) dx = -ln|cos(x)| + c
int cot(x) dx = ln|sin(x)| + c
int cos(x) dx = sin(x) + c
int sin(x) dx = -cos(x) + c
int sqrt(a^2-u^2) dx = sin^-1(u/a) + c
int 1/(a^2+u^2) dx = 1/a * tan^-1(u/a) + c
int ln(x) dx = x ln(x) - x + c

U-Substitution

Let u = f(x) (can be more than one variable)
Determine: du = f'(x) dx and solve for dx.
Then, if a definite integral, substitute the bounds for u = f(x) at each bound
Solve the integral using u.

Integration by Parts

int u dv = uv - int v du

Fns and Identities

sin(cos^-1(x)) = sqrt(1-x^2)
cos(sin^-1(x)) = sqrt(1-x^2)

Directional Derivatives

Let z=f(x,y) be a function, (a,b) apoint in the domain (a valid input point) and u = a unit vector (2D)
The Directional Derivative is then the derivative at the point (a,b) in the direction of u or:
D_u f(a,b) = u . grad f(a,b)
(This will return a scalar, 4-D version:
D_u f(a,b,c) = u . grad f(a,b,c)

Tangent Planes

Let F(x,y,z) = k be a surface and P = (x_0,y_0,z_0) be a point on that surface.
Equation of a Tangent Plane:
grad F(x_0,y_0,z_0) . <x-x_0,y-y_0,z-z_0>

Approximations

let z = f(x,y) be a differentiable function
total differential of f = dz
dz = grad f . <dx,dy>
This is the approximate change in z
The actual change in z is the difference in z values:
delta z = z_2 - z_1

Maxima and Minima

Internal Points

- 1. Take the Partial Derivatives with respect to X and Y (f_x and f_y) (Can use gradient)
- 2. Set derivatives equal to 0 and use to solve system of equations for x and y
- 3. Plug back into original equation for z. Use Second Derivative Test for whether points are local max, min, or saddle

Second Partial Derivative Test

- 1. Find all (x,y) points such that grad f(x,y) = 0
- 2. Let D = f_xx(x,y)f_yy(x,y) - f_xy^2(x,y) IF (a) D > 0 AND f_xx < 0, f(x,y) is local max value
(b) D > 0 AND f_xx(x,y) > 0 f(x,y) is local min value
(c) D < 0, (x,y,f(x,y)) is a saddle point
(d) D = 0, test is inconclusive
3. Determine if any boundary point gives min or max. Typically, we have to parametrize boundary and then reduce to a Calc 1 type of min/max problem to solve.
The following only apply only if a boundary is given
1. check the corner points
2. Check each line (0 <= x <= 5 would give x=0 and x=5)
On Bounded Equations, this is the global min and max..second derivative test is not needed.

Lagrange Multipliers

Given a function f(x,y) with a constraint g(x,y), solve the following system of equations to find the max and min points on the constraint (NOTE: may need to also find internal points.):
grad f = lambda grad g
g(x,y) = 0 (or k if given)

sec(tan^-1(x)) = sqrt(1+x^2)
tan(sec^-1(x)) = (sqrt(x^2+1) if x >= 1)
= (-sqrt(x^2+1) if x < -1)
sinh^-1(x) = ln x + sqrt(x^2+1)
sinh^-1(x) = ln x + sqrt(x^2-1), x >= 1
tanh^-1(x) = 1/2 ln x + 1/(1-x^2), -1 < x < 1
sech^-1(x) = ln[1+sqrt(1-x^2)]/x, 0 < x <= 1
sinh(x) = (e^x - e^-x)/2
cosh(x) = (e^x + e^-x)/2

Trig Identities

sin^2(x) + cos^2(x) = 1
1 + tan^2(x) = sec^2(x)
1 + cot^2(x) = csc^2(x)
sin(x +/- y) = sin(x)cos(y) +/- cos(x)sin(y)
cos(x +/- y) = cos(x)cos(y) +/- sin(x)sin(y)
tan(x +/- y) = (tan(x) +/- tan(y))/(1 +/- tan(x)tan(y))
sin(2x) = 2sin(x)cos(x)
cos(2x) = cos^2(x) - sin^2(x)
cosh^2(x) - sinh^2(x) = 1
1 + tan^2(x) = sec^2(x)
1 + cot^2(x) = csc^2(x)
sin^2(x) = 1 - cos^2(x)
cos^2(x) = 1 - sin^2(x)
tan^2(x) = 1/cos^2(x) - 1
sin(-x) = -sin(x)
cos(-x) = cos(x)
tan(-x) = -tan(x)

Calculus 3 Concepts

Cartesian coords in 3D
given two points:
(x_1,y_1,z_1) and (x_2,y_2,z_2),
Distance between them:
sqrt((x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2)
Midpoint:
((x_1+x_2)/2, (y_1+y_2)/2, (z_1+z_2)/2)
Sphere with center (h,k,l) and radius r:
(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2

Vectors

Vector: u
Unit Vector: u-hat
Magnitude: ||u|| = sqrt(u_1^2 + u_2^2 + u_3^2)
Unit Vector: u-hat = u/||u||

Dot Product

u . v
Produces a Scalar
(Geometrically, the dot product is a vector projection)
u-hat . u_1, u_2, u_3 >= 0
v-hat . v_1, v_2, v_3 >= 0
u-hat . v-hat = 0 means the two vectors are Perpendicular
theta is the angle between them.
u-hat . v-hat = ||u|| ||v|| cos(theta)
u-hat . v-hat = u_1 v_1 + u_2 v_2 + u_3 v_3
NOTE:
u-hat . v-hat = cos(theta)
||u||^2 = u-hat . u-hat
u-hat . v-hat = 0 when u perp v
Angle Between u-hat and v-hat:
theta = cos^-1((u-hat . v-hat)/(||u|| ||v||))

Double Integrals

With Respect to the xy-plane, if taking an integral over a region R in the xy-plane,
int double int_R f(x,y) dy dx = int double int_R f(x,y) dx dy
if f(x,y) is cutting in vertical rectangles,
int double int_R f(x,y) dx dy = int double int_R f(x,y) dy dx
if f(x,y) is cutting in horizontal rectangles,
int double int_R f(x,y) dy dx = int double int_R f(x,y) dx dy

Polar Coordinates

When using polar coordinates,
dA = r dr dtheta

Surface Area of a Curve

let z = f(x,y) be continuous over S (a closed Region in 2D domain)
Then the surface area of z = f(x,y) over S is:
SA = int double int_S sqrt(f_x^2 + f_y^2 + 1) dA

Triple Integrals

int triple int_V f(x,y,z) dv = int triple int_V f(x,y,z) dz dy dx
Note: dv can be exchanged for dx dy dz in any order, but you must then choose your limits of integration according to that order

Jacobian Method

int double int_R f(g(u,v), h(u,v)) |J(u,v)| du dv = int double int_R f(x,y) dx dy

J(u,v) = |partial g/partial u partial g/partial v; partial h/partial u partial h/partial v|

Common Jacobians:
Rect. to Cylindrical: r
Rect. to Spherical: rho^2 sin(phi)

Vector Fields

let f(x,y,z) be a scalar field and F(x,y,z) = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k be a vector field,
Divergence of F:
div F = grad f . F = partial M/partial x + partial N/partial y + partial P/partial z
Curl of F:
curl F = partial N/partial x - partial M/partial y; partial P/partial x - partial M/partial y; partial N/partial y - partial M/partial x

Line Integrals

C given by x = x(t), y = y(t), t in [a,b]
int_C f(x,y) ds = int_a^b f(x(t), y(t)) |v(t)| dt
where ds = sqrt((dx/dt)^2 + (dy/dt)^2) dt
or sqrt(1 + (dy/dx)^2) dx
To evaluate a Line Integral,
get a parametrized version of the line (usually in terms of t, though in exclusive terms of x or y is ok)
-evaluate for the derivatives needed (usually dy, dx, and/or dt)
-plug in to original equation to get in terms of the independent variable
-solve integral

Projection of u onto v:
proj_v u = (u . v / ||v||^2) v

Cross Product

u x v
Produces a Vector
(Geometrically, the cross product is the area of a parallelogram with sides ||u|| and ||v||)
u x v = <u_1, u_2, u_3> x <v_1, v_2, v_3> = <u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1>

Lines and Planes

Equation of a Plane
(x_0, y_0, z_0) is a point on the plane and <A, B, C> is a normal vector

A(x-x_0) + B(y-y_0) + C(z-z_0) = 0
A, B, C >= 0
Ax + By + Cz = D where D = Ax_0 + By_0 + Cz_0

Equation of a line

A line requires a Direction Vector
u-hat = <u_1, u_2, u_3> and a point (x_1, y_1, z_1)
then,
a parameterization of a line could be:
x = u_1 t + x_1
y = u_2 t + y_1
z = u_3 t + z_1

Distance from a Point to a Plane

The distance from a point (x_0, y_0, z_0) to a plane Ax+By+Cz=D can be expressed by the formula:
d = |Ax_0 + By_0 + Cz_0 - D| / sqrt(A^2 + B^2 + C^2)

Coord Sys Conv

Cylindrical to Rectangular
x = r cos(theta)
y = r sin(theta)
z = z
Rectangular to Cylindrical
r = sqrt(x^2 + y^2)
tan(theta) = y/x
z = z
Spherical to Rectangular
x = rho sin(phi) cos(theta)
y = rho sin(phi) sin(theta)
z = rho cos(phi)
Rectangular to Spherical
rho = sqrt(x^2 + y^2 + z^2)
tan(theta) = y/x
cos(phi) = z / sqrt(x^2 + y^2 + z^2)
Spherical to Cylindrical
rho = rho sin(phi)
theta = theta
phi = phi
Cylindrical to Spherical
rho = rho cos(phi)
phi = phi
tan(theta) = y/x
cos(phi) = z / sqrt(x^2 + y^2 + z^2)
Spherical to Rectangular
x = rho sin(phi) cos(theta)
y = rho sin(phi) sin(theta)
z = rho cos(phi)

Surfaces

Ellipsoid
x^2/a^2 + y^2/b^2 + z^2/c^2 = 1



Hyperboloid of One Sheet

x^2/a^2 + y^2/b^2 - z^2/c^2 = 1
(Major Axis: z because it follows -)



Hyperboloid of Two Sheets

x^2/a^2 - y^2/b^2 - z^2/c^2 = 1
(Major Axis: Z because it is the one not subtracted)



Elliptic Paraboloid

z = x^2/a^2 + y^2/b^2
(Major Axis: z because it is the variable NOT squared)



Hyperbolic Paraboloid

z = x^2/a^2 - y^2/b^2
(Major Axis: Z axis because it is not squared)



Elliptic Cone

(Major Axis: Z axis because it's the only one being subtracted)
x^2/a^2 + y^2/b^2 - z^2/c^2 = 0



Cylinder

1 of the variables is missing
OR
(x-a)^2 + (y-b)^2 = c
(Major Axis is missing variable)

Partial Derivatives

Partial Derivatives are simply holding all other variables constant (and act like constants for the derivative) and only taking the derivative with respect to a given variable.

Surface Integrals

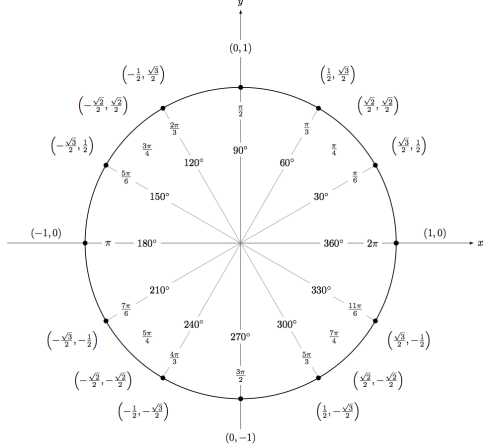
Let R be closed, bounded region in xy-plane
-f be a fn with first order partial derivatives on R
-G be a surface over R given by z = f(x,y)
-g(x,y,z) = g(x,y,f(x,y)) is cont. on R
Then
int double int_R g(x,y,z) dS = int double int_R g(x,y,f(x,y)) dA
where dS = sqrt(f_x^2 + f_y^2 + 1) dy dx

Flux of F across G

int double int_G F . ndS = int double int_G (-M f_x - N f_y + P) dx dy
where:
-F(x,y,z) = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k
-G is surface f(x,y)=z
-n is upward unit normal on G.
-f(x,y) has continuous 1st order partial derivatives

Unit Circle

(cos, sin)



Given z=f(x,y), the partial derivative of z with respect to x is:

f_x(x,y) = z_x = partial z / partial x = partial f(x,y) / partial x
likewise for partial with respect to y:
f_y(x,y) = z_y = partial z / partial y = partial f(x,y) / partial y

Notation

For f_{xy}, work "inside to outside" f_x then f_{xy}, then f_{xyy}
f_{xyy} = partial^3 f / partial x partial y^2
For partial^3 f / partial x^2 partial y, work right to left in the denominator

Gradients

The Gradient of a function in 2 variables is grad f = <f_x, f_y>
The Gradient of a function in 3 variables is grad f = <f_x, f_y, f_z>

Chain Rule(s)

Take the Partial derivative with respect to the first-order variables of the function times the partial (or normal) derivative of the first-order variable to the ultimate variable you are looking for summed with the same process for other first-order variables this makes sense here. Example:

let x = x(s,t), y = y(t) and z = z(x,y).
z then has first partial derivative:
partial z / partial s and partial z / partial t
Note: the use of "d" instead of "partial" with the function of only one independent variable

In this case (with z containing x and y as well as x and y both containing s and t), the chain rule for partial z / partial s = partial z / partial x partial x / partial s + partial z / partial y partial y / partial s
The chain rule for partial z / partial t = partial z / partial x partial x / partial t + partial z / partial y partial y / partial t

Note: the use of "d" instead of "partial" with the function of only one independent variable

Limits and Continuity

Limits in 2 or more variables
Limits taken over a vectorized limit just evaluate separately for each component of the limit.

Strategies to show limit exists

- 1. Plug in Numbers, Everything is Fine
- 2. Algebraic Manipulation
-factoring/dividing out
-use trig identities
- 3. Change to polar coords
if f(x,y) -> (0,0) as r -> 0
Strategies to show limit DNE
1. Show limit is different if approached from different paths
(x=y, x=y^2, etc.)
2. Switch to Polar coords and show the limit DNE.
Continuity
A fn, z = f(x,y), is continuous at (a,b) if
f(a,b) = lim_{(x,y)->(a,b)} f(x,y)
Which means:
1. The limit exists
2. The fn value is defined
3. They are the same value

Other Information

z/r = sqrt(x^2 + y^2)
Where a Cone is defined as z = sqrt(a(x^2 + y^2))
In Spherical Coordinates,
phi = cos^-1(z / sqrt(x^2 + y^2 + z^2))
Right Circular Cylinder:
V = pi r^2 h, SA = pi r^2 + 2 pi r h
lim_{n->inf} (1 + 1/n)^n = e
Law of Cosines:
a^2 = b^2 + c^2 - 2bc cos(theta)

Stokes Theorem

Let:
-S be a 3D surface
-F(x,y,z) = M(x,y,z)i + N(x,y,z)j + P(x,y,z)k
-M,N,P have continuous 1st order partial derivatives
-C piece-wise smooth, simple, closed, curve, positively oriented
-T is unit tangent vector to C.
Then,
int_C F . dS = int double int_S (grad F x F) . ndS = int double int_S (f_x i + f_y j + f_z k) . (n_x i + n_y j + n_z k) dA
Remember:
f . T ds = int_C (M dx + N dy + P dz)